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ELEMENTARY APPLIED MECHANICS.

ELEMENTARY

APPLIED MECHANICS.

BEING

THE SIMPLER AND MORE PRACTICAL CASES OF STRESS AND STRAIN WROUGHT OUT INDIVIDUALLY FROM FIRST PRINCIPLES BY MEANS OF ELEMENTARY MATHEMATICS, ILLUSTRATED BY DIAGRAMS AND GRADUATED EXAMPLES, INTENDED AS AN EASY INTRODUCTION TO THE GENERAL TREATMENT OF THE SUBJECT IN RANKINE'S APPLIED MECHANICS.

BY

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PART FIRST.

ELASTICITY, RESILIENCE.

PURE STRAIN, SIMPLE AND COMPOUND

THE ELLIPSE OF STRESS.

APPLICATION TO EARTHWORK.

PREFACE.

THIS volume is the first of two on the subject of Internal Stress and Strain which are intended to supply the felt want of a complete and systematic set of exercises. It is based on the late PROFESSOR RANKINE'S treatment of the subject in his *Applied Mechanics* and *Civil Engineering*, and will serve as a companion, or as an introduction, to these volumes.

22nd July, 1879.

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ELEMENTARY

APPLIED MECHANICS.

STRESS.

ELASTICITY is a property of matter. When dealing with the equilibrium of a body under the action of external forces, in order to find the relations among those external forces, the matter of the body is considered to be perfectly rigid, or, in other words, to have no elasticity. When external forces, the simplest of which are stresses acting really on a part of the surface of a body, are considered to act at points on the surface, it is taken for granted that the matter of the body is infinitely strong at such points. But after considering the equilibrium of the body as a whole, we may consider the equilibrium of all or any of its parts. If we take a part on which an external stress directly acts, equilibrium is maintained between that external stress acting on the free surface and the components parallel to it, of stresses which the cut surface of the remaining part exerts on its cut surface.

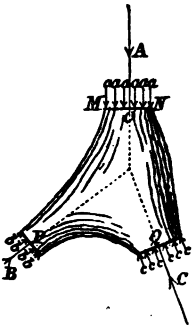


Fig. 1.

Let MONQP be a solid in equilibrium under the action of the three external uniform stresses acting on planes of its surface at O, P, and Q. Let MN be the trace of the plane at O under the uniform stress A. The stresses *aaa*...*bbb*...*ccc* may be represented in amount and direction by the single forces A, B, and C acting at the points O, P, and Q, rigidly connected. We know that, by the triangle of forces, A, B, and C are proportional to the sides of a triangle DFE drawn with its sides parallel to their directions. Also that they are in one plane and meet at one point. Hence we infer that the stresses which they represent are all parallel to the plane of the paper, and that the planes of action of b and c are at right angles to the plane of the paper as well as that of a. Thus we find the relation among the external forces.

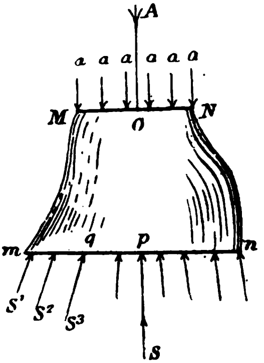


Fig. 2.

Let a plane mn divide the solid into two parts. Consider the equilibrium of the part MNmn. s_1, s_2, s_3 ... are the stresses exerted at all points of the cut surface of MNmn by the cut surface of the other part. S is the sum of their components parallel to the direction of A, acting through P, the centre of pressure. Because there is equilibrium, S is equal and opposite to A; and they act in one straight line. Also the remaining rectangular components of s_1, s_2, s_3 are themselves in equilibrium. Thus we see there is a stress on the plane mn and know the amount of it in one direction.

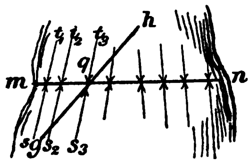


Fig. 3.

Had we been considering the equilibrium of the other part of the solid, the stresses on mn would have been acting on the other surface as $t_1, t_2, t_3, \dots$ in opposite directions to $s_1, s_2, s_3, \dots$ and of equal intensities. Thus on the plane mn there are pairs of actions, acting at all points of it, as s_3, t_3, at q. These vary in intensity and obliquity to mn at different points of the plane. If another plane, as gh, dividing the solid, pass through q, there will be, *similarly, pairs of actions* at all points of it, and *a pair* of definite intensity and direction at the point q. If we know the stress at the point q in intensity and direction on all planes passing through q, we are said to know *the internal stress* at the point q of the solid. Similarly for all points of the solid.

The pairs of actions as s_3, t_3 act respectively on the cut surfaces of the upper and under parts of the solid; but *mn* may be considered to be a *thin layer* of the solid with s_3 and t_3 acting on its under and upper surfaces. It must be, however, infinitely thin, this layer of the solid; otherwise its two surfaces would be different sections of the solid and s_3 and t_3 not necessarily equal and opposite. If *gh* be also considered a thin layer, and H and K be the pair of actions on it at the point q on the two sides of it, then will the point q be a solid, in figure a parallelopiped, with a pair of stresses acting upon its opposite pairs of faces. s_3 and t_3 being equal, S is now put for each, and H is put for both H and K; and since q, instead of being a point in both planes, has small surfaces in both, though so infinitely small that the stresses over them do not vary from the intensities at the point q, yet surfaces, the stresses spread over which it is more convenient to represent by *sets of equal arrows* S S S ..., H H H ....

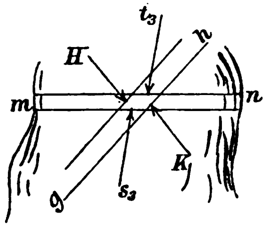


Fig. 4.

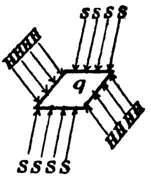


Fig. 5.

In this way of representing the *internal stress* at q the parallelopiped is all of the solid to be considered. It must be borne in mind, however, that q is indefinitely thin each way, and might be more accurately exhibited thus; which must be understood to be such small portions of the planes *mn* and *gh* that the stress remains constant in direction and intensity throughout their extent.

State of Simple Strain.

Thus we see that in a solid acted upon by external forces, every particle exerts stress upon all those surrounding it. Such a body is said to be in *a state of strain.* In solids the phenomenon is marked by an alteration of shape, but not necessarily of bulk.

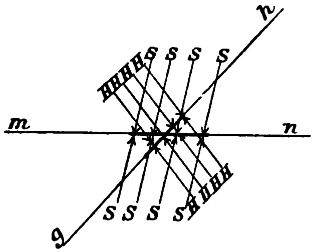


Fig. 6.

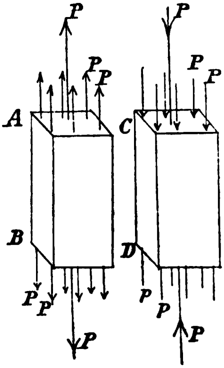


Fig. 7.

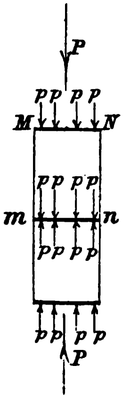


Fig. 8.

Let A B and C D be acted upon by two equal and opposite forces P and P in the direction of their length acting in A B away from each other, and in C D towards each other. If P be uniformly distributed over the area A, the section of A B perpendicular to its length, the intensity of the stress on it is

$$p = \frac{P}{A}.$$

Let the prism be of unit thickness normal to the paper; then will the line M N be equal to the area of the section of the prism perpendicular to its axis and

$$p = \frac{P}{MN}.$$

At any internal layer mn, perpendicular to the axis of the prism, the intensity is also p, for the equilibrium of the parts requires it; and not only is the stress of this same intensity at all points of one such section, but also upon all such sections. The solids A B and C D are said to be in *a state of simple strain*, in the case of A B of *extension*, and in that of C D of *compression*. It is usual to consider the first as positive and the second as negative.

The change of dimensions due to a simple state of strain is an alteration of the length of the solid in the direction of the stress with or without an accompanying alteration of its other dimensions. Thus a piece of cork in a state of simple compression has become shorter in the direction of the thrust, yet with scarcely any, certainly without a corresponding increase of area, normal to the thrust. Again, a piece of indiarubber grows shorter in the direction of the thrust with an almost exactly proportionate increase of area normal to it.

The increase of length in the case of an extension is the *augmentation*, in that of a compression it is a negative augmentation, and in either case it is the amount of strain. The measure of the strain is the ratio of the augmentation to the original unstrained length.

DEFINITION. Longitudinal strain $= \dfrac{\text{augmentation of length}}{\text{length}}$

where both are in the same name, that is, both in inches or feet, &c. The foot being the unit of length, it is most convenient to take both in feet; then

$$\text{longitudinal strain} = \frac{\text{augmentation in feet}}{\text{length in feet}}.$$

Suppose the denominator on the right-hand side of the equation to be unity, then

$$\begin{aligned}\text{longitudinal strain} &= \text{augmentation in feet of 1 foot of the substance.}\\ &= \text{augmentation per foot of length, expressed in feet.}\end{aligned}$$

Hence the total augmentation or *amount* of strain in feet equals *the length in* feet multiplied by the strain.

If the augmentation equal the length, that is, if the piece be stretched to double its original length or compressed to nothing, then from the definition

$$\text{strain} = \text{unity}.$$

Examples.

In the following questions the weight of the material is neglected.

1. A tie rod in a roof, whose length is 142 feet, stretches 1 inch when bearing its proper stress. What strain is it subjected to?

$$\text{augmentation} = 1 \text{ in.}$$
$$\text{unstrained length} = 1704 \text{ in.}$$

$$\text{strain} = \frac{\text{augmentation}}{\text{length}} = \frac{1}{1704} \text{ or } \cdot 0006.$$

2. How much will a tie rod 100 feet long stretch when subjected to ·001 of strain?

$$\frac{\text{augmentation}}{\text{length}} = \text{strain}.$$

$$\therefore \text{augmentation} = \text{strain} \times \text{length} = \cdot 001 \times 100 \text{ ft.} = \cdot 1 \text{ ft.}$$

3. A cast-iron pillar 18 feet high shrinks to 17·99 feet when loaded. What is the strain?

$$\text{augmentation of length} = - \cdot 01 \text{ ft.}$$

$$\text{strain} = \frac{\text{augmentation}}{\text{length}} = \frac{-\cdot 01 \text{ ft.}}{18 \text{ ft.}} = -\frac{1}{1800} \text{ or } -\cdot 0005.$$

4. Two wire cables, whose lengths are 100 and 90 fathoms respectively, while mooring a ship are stretched, the first 3 inches and the second 2·75 inches. What strains do they sustain? Which sustains the greater? Give the ratio of the strains.

For the 100-fathom cable

$$\textit{strain} = \frac{\textit{augmentation}}{\textit{length}} = \frac{3 \text{ in.}}{7200 \text{ in.}} = \cdot 000417.$$

For the 90 fathom cable

$$\text{strain} = \frac{\text{augmentation}}{\text{length}} = \frac{2{\cdot}75 \text{ in.}}{6480 \text{ in.}} = {\cdot}000422.$$

The 90-fathom cable is the more strained.
Ratio of these strains is 417 to 422.

5. A 30-feet suspension rod stretches $\frac{1}{20}$ inch under its load. Find the strain upon it.

$$\text{strain} = {\cdot}00014.$$

6. How much does another of them, which is 23 feet long, stretch when equally strained?

$$\text{augmentation} = {\cdot}039 \text{ inches}.$$

7. A submarine cable is tested with a strain of ·0002. How much did it stretch per 100 fathoms?

$$\text{aug. per 100 fathoms} = 1{\cdot}44 \text{ in.}$$

8. What is the strain upon a wooden strut 60 feet long when compressed to 59·97 feet?

$$\text{strain} = - {\cdot}0005.$$

9. A violin string 10 inches long is stretched to $10\frac{1}{4}$ inches. What is the strain upon it?

$$\text{strain} = \frac{1}{40} = {\cdot}025.$$

10. An indiarubber string 6 inches in length is stretched till it is a foot long. Find the strain.

$$\text{strain} = 1.$$

11. A cast-iron pillar bears a strain of ·001; its original length was 10 feet. Find its altered length.

$$\begin{aligned}\text{augmentation} &= - {\cdot}12 \text{ inches,}\\ \text{altered length} &= 119{\cdot}88 \quad ,,\end{aligned}$$

12. A pillar 40 feet high, designed to prop up a beam already supported at the ends, fits exactly into its place. If the greatest strain to which it is safe to subject the pillar be ·0004, what thickness of wedge ought to be driven between *the beam and* its top?

The thickness is the same as the contraction which the pillar must undergo to produce the necessary strain.

$$\text{thickness} = -\ \text{aug.} = \cdot 192 \text{ in.}$$

This is upon the supposition that the beam does not rise up when the wedge is driven.

ELASTICITY.

The *Elasticity* of a solid is the tendency it has when strained to regain its original shape. If two equal and similar prisms of different matter be strained similarly and to an equal degree, that which required the greater stress is the more elastic—*e.g.*, a copper wire 1,000 inches long was stretched an inch by a weight of 680 lbs., while an iron wire of the same section and length required 1,000 lbs. to stretch it an inch. Hence iron is more elastic than copper. If they be strained by equal stresses, that which is the more strained is the less elastic—*e.g.*, the same copper wire is stretched as before 1 inch by a weight of 680 lbs., while the iron one is only stretched a $\frac{17}{25}$th part of an inch by 680 lbs.

Hence the elasticities of different substances are proportional to the stresses applied, and inversely proportional to the accompanying strains.

If similar rods of steel and indiarubber be subjected to the same stress, the indiarubber experiences an immensely greater strain, so that steel is very much more elastic than indiarubber.

If two similar rods of the same matter, or the one rod successively, be strained by different stresses, the corresponding strains are proportional to the stresses. Thus, if a 480 lb. stress stretch a copper wire one inch, then a 960 lb. stress will stretch it, or a similar rod, two inches.

Hooke's Law is "*The strain is proportional to the stress.*" It amounts to—"the effect is proportional to the cause." It is only true for solids within certain limits:—*e.g.*, 2,400 lbs. should stretch the copper wire mentioned above five inches by Hooke's law, but it would really tear it to pieces; *and although 1,920* lbs. applied very gradually will not tear *it, yet it will stretch it* more than four inches; and further, when that stress is removed the wire will not contract to its original length again. Strain and stress are mutually cause and effect. The effect of stress upon a solid is to produce strain; and, conversely, a body in a state of strain exerts stress. The expressions "Strain due to the stress," &c., and "Stress due to the strain," &c., are both correct.

If a solid be strained beyond a certain degree, called the proof strain, it does not regain its original length when released from the strain; in such a case the permanent alteration of length is called a *set*.

DEF. *The Proof Load* is the stress of greatest intensity which will just produce a strain having the same ratio to itself which the strains bear constantly to the stresses producing them for all stresses of less intensity.

If a stress be applied of very much greater intensity, the piece will break at once; if of moderately greater intensity, the piece will take a set; and although only of a little greater intensity, yet if applied for a long time the piece will ultimately take a set; and if it be applied and removed many times in succession the strain will increase each time and the piece ultimately break. For all stresses of intensities less than the proof load the elasticity is constant for the same substance, and the

$$\text{DEF. Modulus of Elasticity} = \frac{\text{intensity of stress}}{\text{strain due to it}}$$
$$= \text{stress per unit strain.}$$

If the denominator on the right-hand side of the equation be unity, then the numerator is the stress which produces unit strain, and

Mod. of elasticity = stress which would produce unit strain,

on supposition of rod not experiencing a set and Hooke's law holding.

For most substances the proof stress is a mere fraction of *E*. the modulus of elasticity. For steel the proof stress is scarcely $\frac{1}{100000}$th part of *E*. Hence in the equation above the word *would* is employed, as it would be absurd to say that *E* equalled the stress that will produce unit strain, that being *an impossibility* with most substances; and even when possible, as in the case of indiarubber, the strains at such a pitch will have ceased to be proportional to the stresses producing them, and hence E will be no longer of a constant value. But the definition is quite accurate and definite for all substances amounting to this, that for any substance

$E = 10$ times the stress that will produce a strain of $\frac{1}{10}$th,

if such a pitch of strain be possible and within the limit of strain, that is, not greater than the proof strain.

But if not, then,

$E = 100$ times the stress that will produce a strain of $\frac{1}{100}$th,

if such a pitch of strain be possible and within the limits of strain, that is, not greater than the proof strain.

Thus for steel E equals one million times the stress which will produce a strain of one millionth part. *Pliability* is a term applied to the property which indiarubber possesses in a higher degree than steel.

Examples.

13. A wrought-iron tie-rod has a stress of 18,000 lbs. per square inch of section which produces a strain of ·0006. Find the modulus of elasticity of the iron.

$$E = \frac{\text{intensity of stress}}{\text{strain}} = \frac{18000}{\cdot 0006}$$
$$= 30{,}000{,}000 \text{ lbs. per square inch.}$$

14. A tie-rod 100 feet long has a sectional area of 2 square inches, it bears a tension of 32,000 lbs, by which it is stretched $\frac{3}{4}$ths of an inch. Find the intensity of the stress, the strain, and modulus of elasticity.

$$\text{stress} = \frac{\text{total stress}}{\text{area}} = \frac{32{,}000 \text{ lbs.}}{2 \text{ sq. in.}} = 16{,}000 \text{ lbs. per sq. in.}$$

$$\text{strain} = \frac{\text{aug. of length}}{\text{length}} = \frac{\cdot 75 \text{ in.}}{1200 \text{ in.}} = \cdot 000625.$$

$$E = \frac{\text{stress}}{\textit{strain}} = \frac{16000}{\cdot 000625} = 25{,}600{,}000 \text{ lbs. per sq. in.}$$

15. A cast-iron pillar one square foot in sectional area bears a weight of 2000 tons, what strain will this produce, E for cast iron being 17,000,000 lbs. ?

$$\begin{aligned}
\text{total stress} &= 2000 \text{ tons per sq. foot.} \\
&= 2000 \times 2240 \text{ lbs. per sq. foot.} \\
\text{stress} &= \frac{2000 \times 2240}{144} = 31111{\cdot}\dot{1} \text{ lbs. per sq. in.} \\
E &= \frac{\text{stress}}{\text{strain}}, \\
\text{or } 17{,}000{,}000 &= \frac{31111}{\text{strain}}. \\
\therefore \text{ strain} &= \frac{31111}{17{,}000{,}000} = {\cdot}0018 \text{ ft. per ft. of length.}
\end{aligned}$$

16. The modulus of elasticity of steel is 35,000,000. How much will a steel rod 50 feet long and of $\frac{1}{8}$th inch sectional area be stretched by a weight of one ton ?

$$\begin{aligned}
\text{total stress} &= 2240 \text{ lbs.} \\
\text{stress} &= \frac{\text{total stress in lbs.}}{\text{area in sq. in.}} \\
&= 2240 \div \tfrac{1}{8} = 17{,}920 \text{ lbs. per sq. inch.} \\
E &= \frac{\text{stress}}{\text{strain}}. \\
\therefore \text{ strain} &= \frac{\text{stress}}{E} \\
&= \frac{17{,}920}{35{,}000{,}000} = {\cdot}000512. \\
\frac{\text{elongation}}{\text{length}} &= \text{strain.} \\
\therefore \text{ elongation} &= \text{strain} \times \text{length} \\
&= {\cdot}000512 \times 50 \\
&= {\cdot}0256 \text{ feet or } \tfrac{1}{3} \text{ of an inch.}
\end{aligned}$$

17. An iron wire 600 yds. long and $\frac{1}{80}$th of sq. inch in section, in moving a signal sustains a pull = 250 lbs.; how much will it stretch, assuming $E = 25{,}000{,}000$?

stress = 20,000 lbs. per sq. in.,
strain = ·0008,
elongation = 1·44 ft.

18. Modulus of elasticity of copper is 17,000,000; what weight ought to stretch a copper thread, of 12 inches in length and ·004 inches in sectional area, $\frac{1}{100}$th part of an inch. If after the removal of the weight the thread remains a little stretched, what do you infer about the weight and about the strain to which the thread was subjected?

strain = $\frac{1}{1200}$,
stress = 14167 lbs. per sq. inch,
weight = 56·668 lbs.

Since this weight causes a set it is greater than the proof load.

19. A wooden tie 12 inches × 7 inches and 40 feet long was tested with a pull of 130 tons which stretched it 1·28 inches. Find the modulus of elasticity of the wood.

$$\text{stress} = \frac{291200 \text{ lbs.}}{84 \text{ sq. in.}} = 3466{\cdot}\dot{6} \text{ lbs. per sq. in.}$$

$$\text{strain} = \frac{1{\cdot}28 \text{ in.}}{480 \text{ in.}} = {\cdot}002\dot{6}.$$

$$\therefore\ E = 1{,}300{,}000 \text{ lbs.}$$

20. One rod of an hydraulic hoist is 50 feet long and 1 inch in diameter, it is attached to a plunger 4 inches in diameter upon which the pressure of the water is 800 lbs. per square inch. E being 30,000,000, how much will the rod be compressed, and what is the strain?

strain = – ·000427 ft. per ft.
compression = amount of strain = ·256 in.

21. A glass thread is $\frac{1}{1000}$th of a sq. inch in sectional area, and 15 inches long. What weight would be required to stretch it $\frac{1}{100}$th part of an inch, for glass E being 8,000,000?

stress = 5,333 lbs. per. sq. in.
weight = *amount* of stress = 5·3 lbs.

The Production of Strain.

We have as yet only considered the *statical condition* of strain, *i.e.*, of a body kept in a state of strain by external forces, these forces being balanced by the reactions of the solid at their places of application due to the elasticity, and the forces exerted on any portion of the solid being in equilibrium with the reactions of the contingent parts. Thus when we found that 32,000 lbs. produced a strain of ·00063 on a tie rod 100 feet long and 2 sq. inches in area, in all stretching it ¾ths of an inch, we meant that the weight *kept it* at that strain; the rod is supposed to have arrived at that pitch of strain and to be at rest, to be stretched the ¾th inch, and so (by its elasticity or tendency to regain its original length) to balance the weight. We have taken no notice of the process by which the rod came to the strain, nor do we say it was the weight that brought it to that state, the weight being only a convenient way of giving the value of the stress on the rod when forcibly *kept* strained. In fact an actual weight of 32,000 lbs. is capable of producing greater strains on the rod in question; depending upon how it is applied to the rod as yet unstrained. The weight might be attached by a chain to the end of a rod and let drop from a height; when the chain checked its fall it would produce a strain on the rod at the instant greater the greater the height through which it dropt. Still, if that strain were not greater than the proof strain, the weight upon finally coming to rest after oscillating a while could only *keep* the rod at the strain ·00063.

We come now to consider the *kinetic relations* between the stress and the strain, that is, while the strain is being produced, the matter of the body being then in motion, we are consequently considering the relations among forces acting upon matter in motion.

If a simple stress of a specific amount be applied to a body it produces a certain strain, and in doing so the stress does work, for it acts through a space in the direction of its action equal to the total strain. But if this stress is applied gradually, so as not to produce a shock, its value increases gradually from zero to its full value, and the work it does will be equal to its mean value, multiplied by the space through which it has acted. And since the stress increases in proportion to the elongation, its average value will equal half of its full value. For example, if a stress of 30,000 lbs. be applied to a rod and produce a strain of $\frac{3}{4}$ inch, it will do $\frac{30000}{2} \times \frac{3}{4} = 11{,}250$ inch lbs. of work which will be stored up as potential energy in the stretched rod.

Suppose a scale-pan attached to the top of a strut or bottom of a tie and the other end fixed. Let a weight be put in contact with the pan, but be otherwise supported so as to exert no stress on the piece, and the next instant let it rest all its weight on the piece, then will the weight do work against the resistance offered by the straining of the piece till the weight ceases descending and comes to rest, when the piece will be for an instant at the greatest strain under the circumstances, at a strain greater than the weight can keep it at; the unstraining of the piece will therefore cause the weight to ascend again, doing work against it to the amount that the weight did in descending, and so the weight will return to its first position, then begin to descend again, and so oscillate up and down through an amplitude equal to the augmentation. Owing to other properties of the matter, whereby some of the work is dissipated during each strain and restitution, the amplitude diminishes every oscillation, and the weight will finally settle at the middle of the amplitude.

A weight applied in this manner is called *a live load.* A live load produces, the instant it is applied, an augmentation of length *double* of that which it can maintain, and therefore causes an *instantaneous strain* double the strain due to a stress of the same amount as the load.

Let now a weight W be applied in the following way. Divide W into n equal parts of weight w each. If A be the strain due to a stress of amount W, and a the strain due to a stress w, then

$$W = nw.$$

and by Hooke's law, $A = na.$

Let the first weight w be put into the scale-pan. It will produce a strain $2a$ at once, but the piece will settle at a strain a. Now put on the second weight w. It will produce at once an additional strain $2a$, but only of a additional after the piece settles; there being now a total strain $2a$. Add the third weight w. It also will produce at first an additional strain $2a$, but only of a after the piece settles, giving a total now of $3a$; and so, adding them one by one, there will be a strain of $(n - 1)\,a$ when the second last one has been added and the piece has settled. Now, upon adding the nth weight w, it will at first produce an additional strain $2a$, but only of a after the piece settles, giving then a total strain na or A. Thus we have brought the piece to a pitch of strain A by means of the weight W, and only at the instant of adding the last part (w) of it was the piece strained to $(n + 1)\,a$, or to a more than A. By making the parts more numerous into which we divide W, and so each part lighter and producing a lesser strain per part, we can make the strain a the extent to which the piece is strained beyond A at the instant of adding the last part, as small as we please.

By so applying the load W we can bring the piece to the corresponding strain A without at all straining it beyond that. A weight so applied is called a *dead load.*

A live load therefore produces, at the instant of its application, a strain equal to that due to a dead load of double the amount. In designing, the greatest strain is that for which provision must be made, so that live loads must be doubled in amount, and the strain then reckoned as due to that amount of dead load. The dead load, together with twice the live load, is called the *gross load.*

The weights of a structure and of its pieces are generally dead loads. Stress produced by a screw, as in tightening a tie rod, is a dead load. The pressure of earth or water gradually filled in behind a retaining wall, and of steam got up slowly, of water upon a floating body at rest in it, &c., are all dead loads. The weight of a man, a cart, or a train coming suddenly upon a structure, is a live load, so is the pressure of steam coming suddenly into a vessel, so is a portion of the pressure of water upon a floating body which is rolling or plunging. The pressure upon a plunger used to pump water is a live load, but that on a piston when compressing gas is a dead load, the gas being so elastic itself. A load on a chain ascending or descending a pit is a dead load when moving at a constant speed or at rest, but a live load at the starting, and while the speed is increasing, partly a live and partly a dead load. The stress upon the coupling between two railway carriages is a dead load while the speed is uniform, and if the buffers keep the coupling chain tight, the stress is a live load while starting; but if the buffers do not keep it tight, but allow it to hang in a curve when at rest, then the stress upon it at starting will be greater than a live load.

Examples.

22. An iron rod in a suspension bridge supports of the roadway 2,000 lbs., and when a load of 3 tons passes over it, bears one-fourth part thereof. Find the gross load. If the rod be 20 feet long, and $\frac{3}{4}$ of a square inch in section find the elongation, E being 29,000,000.

dead load = 2000 lbs.
live load = 1680 lbs., equivalent to a dead load of 3360 lbs.

$$\therefore \text{gross load} = 5360 \text{ lbs.}$$

$$\text{stress} = \frac{\text{gross load}}{\text{section}}$$

$$= \frac{5360}{\cdot 75} = 7147 \text{ lbs. per sq. in.}$$

$$E = \frac{\text{stress}}{\text{strain}}.$$

$$\therefore \text{strain} = \frac{\text{stress}}{E}$$

$$= \frac{7147}{29000000} = \cdot 000246$$

$$= \frac{\text{elongation}}{\text{length}}.$$

$$\therefore \text{elongation} = \text{length} \times \text{strain}$$

$$= 20 \times \cdot 000246 = \cdot 00492 \text{ ft.} = \cdot 06 \text{ in.}$$

23. A vertical wrought iron rod 200 feet long has to lift a weight of two tons. Find the area of section, first neglecting its own weight; if the greatest strain to which it is advisable to subject wrought iron be ·0005 and $E =$ 30,000,000.

Let A be the sectional area in sq. in.

live load = 4480 lbs. is equivalent to a deadload of 8960 lbs.

$$\therefore \text{stress} = \frac{3960}{\text{A}}.$$

$$E = \frac{\text{stress}}{\text{strain}}.$$

$$\text{or } 30{,}000{,}000 = \frac{8960}{\text{A} \times \cdot 0005}.$$

$$\therefore A = \frac{8960}{\cdot 0005 \times 30{,}000{,}000}.$$

$$= \cdot 597 \text{ sq. in.}$$

24. Find now the necessary section at top of rod, taking the weight into account, calculated from the section found in last.

200 ft. × ·597 sq. in. gives 1433 cub. in.; reckoned at 480 lbs. per cubic foot gives 398 lbs.

Hence live load = 4480 lbs.

dead load = 398 lbs.

∴ gross load = 9358

$$\text{stress} = \frac{9358}{\text{area}}.$$

$$E = \frac{\text{stress}}{\text{strain}}.$$

$$\text{or } 30{,}000{,}000 = \frac{9358}{\text{area} \times \cdot 0005}.$$

$$\therefore \text{area} = \frac{9358}{\cdot 0005 \times 30{,}000{,}000} = \cdot 62 \text{ sq. in.}$$

The weight of the rod being greater when calculated at this section, a third approximation to the sectional area might be made.

25. Taking now the sectional area at ·62 sq. in. find average strain and elongation.

At lowest point

$$\text{gross load} = 8{,}960 \text{ lbs.},$$

$$\text{stress} = \frac{8{,}960,}{\cdot 62}$$

$$\text{strain} = \frac{8{,}960}{\cdot 62 \times 30{,}000{,}000} = \cdot 00048\,;$$

while strain at highest point is ·0005.

$$\therefore \text{ average strain} = \cdot 000495.$$

$$\text{elongation} = \cdot 00049 \times 200 = \cdot 098 \text{ ft.}$$

$$= 1{\cdot}176 \text{ in.}$$

26. A short hollow cast-iron pillar has a sectional area of 12 sq. in. It is advisable only to strain cast iron to the pitch ·0015. If the pillar supports a dead load of 50 tons; being weight of floor of a railway platform, and loaded waggons pass over it, what amount should such load not exceed? $E = 20{,}000{,}000$.

greatest stress = 30,000 lbs. per sq. in.
gross load = 360,000 lbs.
deduct dead load = 112,000 lbs.
gives a dead load = 248,000 lbs.

The live load must not exceed one half of this.

NOTE. Other considerations limit the strength of the pillar if it be long.

RESILIENCE.

DEF. *The Resilience* of a body is the amount of work required to produce the proof strain. A weight one half the proof stress applied as a live load would produce the proof strain, therefore the work done is this weight multiplied by the elongation at proof strain, the distance which the weight has worked through; or

The resilience of a body = ½ amount of proof stress
× elongation at proof strain.

For comparison among different substances the resilience is *measured* by the resilience of one foot of the substance by one square inch in sectional area.

$$\therefore\ R = \tfrac{1}{2} \text{ proof stress} \times \text{proof strain},$$

R being in foot lbs. when the stress is in lbs. per square inch and the strain in feet.

And now comparing the amount of resilience of different masses of the same substance: if two be of equal sectional area, that which is twice the length of the other has twice the amount of resilience (the elongation being double); also if two be of equal length and one have twice the sectional area of the other, then the amount of its resilience is double (the amount of stress upon it being twice that upon the other). That is, the amounts of the resilience of masses of the same substance are proportional to their volumes. This is true not only for pieces in a state of simple strain with which we are in the meantime occupied, but can be proved to be universally true for those in any state of strain however complex.

For any substance R being the amount of resilience of a prism of that substance one foot long by one square inch in sectional area, it follows from the above that the amount of resilience of a cubic inch of the substance will be $\frac{1}{12}R$ or that of any volume will be $\frac{1}{12}R \times$ volume in cubic inches.

The *resilience of* a piece, as defined, is the greatest amount of work which can be done against the elasticity of the piece, without injuring its material.

We can find the amounts of work done upon a piece in bringing it to pitches of strain lower than the proof strain. For brevity we will call this also resilience. Thus, for a piece 1 foot long by 1 square inch in section

amount of resilence $= \frac{1}{2}$ stress × strain is pro. to $(\text{stress})^2$, the strain being proportional to the stress, hence

$$\frac{\text{amount of resilience for any stress}}{\text{the resilience}} = \frac{(\text{stress})^2}{(\text{proof stress})^2}.$$

$$\therefore \text{ amount of resilience} \quad = R \times \left(\frac{\text{stress}}{\text{p. stress}}\right)^2.$$

For a piece of volume V cubic inches, at any stress we have either—

$$\text{amount of resilience} = \tfrac{1}{2} \text{ stress} \times \text{strain} \times \frac{V}{12},$$

$$\text{or} = \tfrac{1}{2} \text{ amt. of stress} \times \text{amt. of strain}.$$

The amount of resilience of a piece, at the instant a live load is applied, will be the product of that load and the instantaneous elongation. Let W be a load the elongation due to which is A. If W be applied as a live load, the instantaneous elongation is $2A$, and the

$$\text{amount of resilience due to a live load } W = W \times 2A.$$

If W be applied as a dead load, the amount of resilience is steadily that of the piece elongated to an amount A is the same as what it would be for an instant upon the application of a live load $\frac{W}{2}$, or

$$\text{amount of resilience due to a dead load } W = \frac{W}{2} \times A.$$

Therefore, a live load produces for an instant an amount of resilience four times that produced by an equal dead load.

Examples.

27. A rod of steel 10 feet long and ·5 of a square inch in section, is kept at the proof strain by a tension of 25,000 lbs., the modulus of elasticity for steel being 35,000,000.

Find the resilience of steel, also the amount of resilience of the rod.

$$\text{proof stress} = \frac{25000}{\cdot 5} = 50{,}000 \text{ lbs. per sq. inch.}$$

$$E = \frac{\text{proof stress}}{\text{proof strain}}.$$

$$\therefore \text{ proof strain} = \frac{\text{proof stress}}{E}$$

$$= \frac{50{,}000}{35{,}000{,}000} = \frac{1}{700}$$

$= \cdot 00143$ elongation in ft. per ft. of length.

resilience, $R = \frac{1}{2}$ proof stress × proof strain

$= \frac{1}{2} \times 50{,}000$ lbs. $\times \cdot 00143$ ft.

$= 35 \cdot 75$ ft. lbs. of work per vol. of 1 ft. in length by 1 sq. in. in sectional area.

amt. of res. of rod $= R \times$ (vol. expressed in number of such prisms)

$$= \frac{1}{12} \, . \, R \times \text{vol. in cub. in.}$$

$$= \frac{1}{12} \times 35 \cdot 75 \times 120 \text{ in.} \times \cdot 5 \text{ sq. in.}$$

$= 178 \cdot 75$ foot-lbs. of work.

Otherwise, to find amount of resilience directly,

$$\text{proof strain} = \frac{1}{700},$$

$$\text{total elongation} = \frac{1}{70} \text{ ft.,}$$

amount of stress $= 25{,}000$ lbs.,

amount of resilience $= \frac{1}{2}$ amount of stress × elongation,

$$= \frac{25000}{2} \times \frac{1}{70},$$

$= 178 \cdot 6$ ft.-lbs. of work.

28. A series of experiments were made on bars of wrought iron, and it was found that they took a set when strained to a degree greater than that produced by a stress 20,000 lbs. per square inch, but not when strained to a less degree. At that pitch the strain was ·0006. Find the resilience of this quality of iron.

$$\begin{aligned} \text{proof stress} &= 20{,}000 \text{ lbs. per sq. in.} \\ \text{proof strain} &= \cdot 0006 \text{ ft. per ft. of length.} \\ R &= \tfrac{1}{2} \times 20{,}000 \times \cdot 0006 = 6 \text{ ft.-lbs.} \end{aligned}$$

29. Find how much work it would take to bring a rod, of the above iron, 20 feet long and 2 square inches in sectional area, to the proof strain.

$$\text{volume} = 480 \text{ cub. in.}$$

R feet-lbs. of work brings to proof strain a rod 1 feet long by 1 square inch in area: that is, of volume 12 cubic inches, and amounts of resilience being proportional to the volumes.

$$\begin{aligned} \text{work required} &= \tfrac{1}{12} R \,.\, \text{vol. in cub. in.} \\ &= \tfrac{1}{12} \times 6 \times 480 = 240 \text{ ft.-lbs.} \end{aligned}$$

or $$\frac{\text{amount of res.}}{R} = \frac{480 \text{ cub. in.}}{12 \text{ cub. in.}} = 40.$$

$$\therefore \ \text{amount of res.} = R \times 40 = 6 \times 40 = 240 \text{ ft.-lbs.}$$

30. A wooden strut 18 square inches in section, and 12 feet long, sustains a stress of 1,000 lbs. per square inch. Find the amount of resilience of the strut, *E* being 1,200,000 lbs.

$$\begin{aligned} \text{half of total stress} &= 9000 \text{ lbs.} \\ \text{elongation} &= \cdot 01 \text{ ft.} \\ \text{amount of resilience} &= 90 \text{ ft.-lbs.} \end{aligned}$$

31. Steam at a tension of 600 lbs. on the square inch is *admitted suddenly* upon a piston 18 inches in diameter. *If the piston rod* be two inches in diameter and 7 feet

long, what is the amount of its resilience at the instant? $E = 30,000,000$.

for live load, stress = 97200 lbs. per sq. in.
gives instant strain = ·00324 ft. per ft. of length.
elongation = ·02268 ft.
resilience of rod = live load × elongation.
= 48600π lbs. × ·02268 ft.
= 1102π ft.-lbs.

32. The chain of a crane is 30 feet long and has a sectional area equivalent to $\frac{1}{2}$ of a square inch, what is the amount of its resilience when a stone of 1 ton weight resting on a wooden frame is lifted by the action of the crane? $E = 30,000,000$.

stress = 4480 lbs. per sq. in.
strain = ·000149.
amount of stress = 2240 lbs.
elongation = ·00447 ft.
resilience of chain = $\frac{1}{2}$ amount of stress × elongation.
= 5 ft.-lbs.

33. If the chain be just tight, but supporting none of the weight of stone, and if now the wooden frame suddenly gives way, what is the amount of resilience of the chain at the instant?

Being now a live load, there is an instantaneous strain of double the former amount.

instantaneous strain = ·000298.
„ elongation = ·00894 ft.
resilience of chain = live load × elongation.
= 2240 × ·00894.
= 20 ft.-lbs.

34. The wire for moving a signal 600 yards distant has, when the signal is down, a tension upon it of 250 lbs., which is maintained by the back weight of the hand lever; under

the circumstances the wire is stretched 1·44 feet (see example 17), and the back weight of the signal, which is 280 lbs., rests portion of its weight upon its bed. The hand lever is suddenly pulled back and locks: the wire being more intensely strained, the signal is raised by the elasticity of the wire partially unstraining. If the point where the wire is attached to the signal moves through ·2 feet, find the range of the point where the wire is attached to the hand lever, also the force which must be exerted there.

When the signal settles up the amount of stress on the wire is 280 lbs.

$$\frac{\text{elongation for 280 lbs.}}{\text{elongation for 250 lbs.}} = \frac{280}{250},$$

$$\text{elongation for 280 lbs.} = \frac{280}{250} \times 1{\cdot}44 = 1{\cdot}613,$$

$$\text{additional elongation} = {\cdot}173 \text{ ft.},$$

$$\therefore \text{range of point at lever} = \text{range of point at signal plus this additional elongation.}$$

$$= {\cdot}2 + {\cdot}173$$

$$= {\cdot}373 \text{ feet.}$$

Thus, when the lever is put back there is upon the wire for an instant before the signal rises an additional elongation of ·373 feet. Hence the tension on the wire the instant the lever is put back will be that due to an elongation of

$$(1{\cdot}44 + {\cdot}373) \text{ ft.} = \frac{1{\cdot}44 + {\cdot}373}{1{\cdot}44} 250 \text{ lbs.}$$

$$= 314{\cdot}8 \text{ lbs.}$$

This is the force which must be exerted at the point where the wire is attached to the hand lever. That is, the instantaneous value of the force used to raise the signal is 34·8 lbs. greater than its weight.

35. On a chain 30 feet long ¾ of a square inch sectional area and having a modulus of elasticity of 25,000,000 lbs. *there is a dead load of* 3,900 lbs. and a live load of 900 lbs.

Find the amount of resilience of chain when dead load only is on, also at instant live load comes on.

$$E = \frac{\text{stress}}{\text{strain}}.$$

$$\therefore \text{strain} = \frac{\text{stress}}{E} = \frac{5200}{25{,}000{,}000} = \cdot 0002,$$

$$\text{elongation} = \cdot 006 \text{ ft.},$$

$$\left.\begin{array}{c}\text{amount of resilience}\\ \text{for dead load}\end{array}\right\} = \tfrac{1}{2} \text{ amount of stress} \times \text{elongation}$$

$$= \tfrac{1}{2} \times 3900 \times \cdot 006$$

$$= 11 \cdot 7 \text{ ft.-lbs.}$$

Live load gives an additional elongation equal to that for a dead load of 1,800 lbs.

$$\frac{\text{elongation (due to 1800 lbs.)}}{\cdot 006 \text{ ft.}} = \frac{1800}{3900}.$$

$$\text{inst. elongation for live load} = \frac{1800}{3900} \times \cdot 006$$

$$= \cdot 00277 \text{ ft.}$$

Now both the 3900 lbs. and the 900 lbs. worked through this ·00277 ft.

$\therefore$ additional resilience = 4800 lbs. × ·00277 ft. = 13·3 ft.-lbs.

amount of resilience at instant live load comes on = 25 ft.-lbs.

36. A rod 20 feet long and ½ inch in sectional area bears a dead load of 5000 lbs. Find the live load which would produce an instantaneous elongation of another $\frac{1}{10}$th inch. $E = 30{,}000{,}000$.

Ans. 3125 lbs.

37. A rod of iron 1 square inch in section and 24 feet long checks a weight of 36 lbs. which drops through 10 feet before beginning to strain it. If $E = 25\,000{,}000$, find greatest strain.

Let $p =$ the stress at instant of greatest strain, then

$$\text{strain} = \frac{p}{E},$$

$$\text{elongation} = \frac{24p}{E},$$

$$\text{amt. of resilience} = \tfrac{1}{2}\text{ amt. of stress} \times \text{elongation}$$

$$= \frac{p}{2} \times \frac{24p}{E} = 12\frac{p^2}{E} \text{ ft.-lbs.}$$

$$\text{Work done by wt. in falling} = 36 \text{ lbs.} \times \left(10 + \frac{24p}{E}\right) \text{ ft.}$$

$$= 360 + \frac{864p}{E} \text{ ft.-lbs.}$$

$$\text{Equating, } \frac{12p^2}{E} = \frac{864p}{E} + 360,$$

$$p^2 - 72p = 30E,$$

$$p^2 - 72p + (36)^2 = 750{,}001{,}296,$$

$$p - 36 = 27386,$$

$$p = 27422 \text{ lbs. per sq. in.}$$

$$\therefore \text{ strain} = \cdot 001097 \text{ ft. per ft. of length.}$$

38. If the weight in No. 37 had fallen through the 10 feet by the time it came first to rest and $E = 30{,}000{,}000$ lbs., what is the greatest strain?

$$\text{amount of resilience} = 360 \text{ ft.-lbs.,}$$

$$\text{or} \quad \frac{12p^2}{E} = 360,$$

$$\therefore \quad p = 30{,}000 \text{ lbs. per sq. inch,}$$

$$\text{strain} = \cdot 001 \text{ ft. per ft. length.}$$

39. If the proof strain of iron be ·001 what is the shortest *length of* the rod of one sq. inch in sectional area which will *not take a set* when subjected to the shock caused by checking

a weight of 36 lbs. dropped through 10 feet? [$E = 30{,}000{,}000$ lbs. per sq. inch.]

Let x = length in feet.

By hypothesis it comes to the proof strain, hence

$$\begin{aligned} \text{elongation} &= \cdot 001 \times x \text{ ft.} \\ \text{P. stress} &= E \times \text{proof strain} \\ &= 30{,}000 \text{ lbs. per sq. inch.} \\ \text{amount of stress} &= 30{,}000 \text{ lbs.} \\ \text{(inst.) amount of res.} &= \tfrac{1}{2} \text{ amount of stress} \times \text{elongation.} \\ &= 15{,}000 \times \frac{x}{1000} = 15x \text{ ft.-lbs.} \end{aligned}$$

Equating to work done by weight,

$$\begin{aligned} 15x &= 360 \\ x &= 24 \text{ ft.} \end{aligned}$$

NOTE. This is the shortest rod of iron one square inch in sectional area which will bear the shock. The volume of this rod is 288 cubic inches, and a rod of iron which has 288 cubic inches of volume will just bear the shock; as 48 feet long by ½ square inch in area or 12 feet long by 2 square inches sectional area.

The 10 feet fallen through by the weight includes the elongation of rod. When the question is to find the shortest rod to sustain the shock in the case where the weight falls through 10 feet *before it begins to strain* the rod the volumes of the rods would not be exactly equal for different sectional areas, for a long thin rod will sustain a greater elongation than a short thick one, and as the falling weight works through this elongation over and above the 10 feet the first rod will require a greater cubical volume than the second.

40. Find the shortest length of a rod of steel which will just bear without injury the shock caused by checking a weight of 60 lbs. which falls through 12 feet before beginning to strain the rod. First for a rod of sectional area 2 square

inches, and then for a rod of ¼ square inch sectional area. Given that for steel $E = 30{,}000{,}000$ lbs. and $R = 15$ ft.-lbs.

Let A = sectional area in sq. inches and x = length in feet.

$$\text{proof stress} \times \text{proof strain} = 2R \qquad \text{def.}$$

$$\frac{\text{proof stress}}{\text{proof strain}} = E \qquad \text{def.}$$

Dividing, $$(\text{proof strain})^2 = \frac{2R}{E}.$$

$$\text{proof strain} = \sqrt{\frac{2R}{E}} = \frac{1}{1000}.$$

$$\text{elongation} = \frac{x}{1000} \text{ feet.}$$

Work done by the weight in falling

$$= 60 \text{ lbs.} \times \left(12 + \frac{x}{1000}\right) \text{ft.}$$

$$= 720 + \frac{3}{50} x \text{ ft.-lbs.}$$

Amount of resilience of rod at proof strain

$$= R \times \left(\frac{1}{12} \text{ vol. in cub. in.}\right)$$

$$= R \times (\text{length in ft.} \times \text{sec. area in sq. in.}).$$

$$= 15 \times x \times 2 = 30x \text{ ft.-lbs. } \textit{first.}$$

$$\text{and } 15 - x - \tfrac{1}{4} = \frac{15}{4} x \text{ ,, ,, } \textit{second.}$$

Equating for first case,

$$30x = 720 + \frac{3}{50} x,$$

$$1497x = 36000,$$

$$x = 24{\cdot}05 \text{ ft. length of rod.}$$

Equating for second case,

$$\frac{15}{4} x = 720 + \frac{3}{50} x,$$

$$738x = 144000,$$

$$x = 195{\cdot}12 \text{ ft. length of rod.}$$

For the first case length is 288·6 inches, and sectional area 2 square inches gives

$$\text{volume} = 577{\cdot}2 \text{ cub. inches.}$$

While for second case length is 2341·44 inches, and sectional area $\frac{1}{4}$ square inch giving

$$\text{volume} = 585{\cdot}3 \text{ cub. inches,}$$

which is a little greater. See Note to No. 39.

This may be exhibited generally; putting W, h, and A for the weight, the distance dropped through, and the sectional area,

$$\text{work done by weight} = W\left(h + x\sqrt{\frac{2R}{E}}\right).$$

$$\text{amount of resilience} = R \,.\, x \,.\, A.$$

$$\text{Equating, } RxA = Wh + Wx\sqrt{\frac{2R}{E}}.$$

$$\therefore \quad x\left(RA - W\sqrt{\frac{2R}{E}}\right) = Wh.$$

$$\therefore \quad x = \frac{Wh}{RA - W\sqrt{\frac{2R}{E}}} \text{ ft.}$$

$$\text{vol.} = A \times 12x \text{ cub. inches.}$$

$$= \frac{12AWh}{RA - W\sqrt{\frac{2R}{E}}} = \frac{12Wh}{R - \frac{W}{A}\sqrt{\frac{2R}{E}}},$$

which increases as A decreases.

41. The greatest (*working*) stress which it is safe to apply repeatedly to iron being 10,000 lbs. per square inch, what is the shortest rod of one square inch in section that may be employed to check the fall of a 36 lb. weight through 3 feet raised and let fall constantly?

Length, 64 feet.

42. Certain rods of wrought iron are for the same structure; those which will not bear a proof strain of ·001 are to be rejected. The testing machine consists of a ring weighing 120 lbs., which, falling through distances marked upward on the instrument, catches on a collar screwed on the lower end of the rod to be tested, whose upper end is fixed. The shock producing the instantaneous strain ·001 is repeated several times, when if there be no set produced the rod is passed. From what height on the scale ought the ring to be dropped when testing rods 15 feet long by ½ square inch sectional area; also, how high should the collar be above zero? $E = 30{,}000{,}000$.

$$E = \frac{\text{proof stress}}{\text{proof strain}}.$$

$$\begin{aligned}
\text{proof stress} &= \cdot 001 \times E = 30{,}000 \text{ lbs. per sq. in.,}\\
\text{amount of stress} &= 15{,}000 \text{ lbs.,}\\
\text{elongation} &= 15 \times \cdot 001 = \frac{3}{200} \text{ ft.} = \cdot 015 \text{ ft.,}\\
\text{amount of resilience} &= \tfrac{1}{2} \text{ amount of stress} \times \text{elongation}\\
&= 7{,}500 \times \frac{3}{200} = 112\cdot 5 \text{ ft.-lbs.}
\end{aligned}$$

Let x be reading in ft. on scale; then

$$\text{work in ft. lbs.} = 120x, \text{ and, equating}$$

$$120x = 112\cdot 5,$$

$$x = 0\cdot 94 \text{ ft.}$$

The collar ought to be above zero the amount of the elongation, so that the weight may only have descended through the 0·94 ft. when it first comes to rest.

$$\begin{aligned}
\text{height of collar} &= \text{elongation}\\
&= \cdot 015 \text{ ft.}
\end{aligned}$$

43. Find readings on scale from which ring should be dropped, and at which collar should be adjusted when testing 20 feet rods 1½ square inches in area.

reading for ring, 3·75 ft.
reading for collar, 0·02 ft.

44. Wooden piles, 2 square feet in section, are being driven by a weight of 7,680 lbs. When 20 feet of a pile is above ground, what is the greatest height the weight should be dropped from, if 4,000 lbs. per square inch be the greatest stress it is desirable to put upon the timber? $E = 1{,}200{,}000$. Neglect the fact that the weight falls through the diminution of length of strut.

$$\text{height} = 5 \text{ ft.}$$

45. It is found that the pile sinks 0·2 feet, so that the straining of the timber acts through ·2 feet besides the elongation. From what height may the weight be dropped without fear of injury, considering this?

$$\begin{aligned}
\text{stress} &= 4000 \text{ lbs. per sq. in.} \\
\tfrac{1}{2} \text{ amt. of stress} &= 576000 \text{ lbs.} \\
\text{strain} &= -\frac{1}{300}. \\
\text{augmentation} &= -\frac{1}{15} \times \cdot067. \\
\text{amt. of resilience} &= \tfrac{1}{2} \text{ amt. of stress} \times (\cdot2 + \cdot067) \\
&= 153792 \text{ ft.-lbs.} \\
\text{work} = 7680\,(x + \cdot267) &= 153792 \text{ ft.-lbs.} \\
\therefore\ 7680x &= 153792 - 2050. \\
&= 151742. \\
\therefore\ x &= 19{\cdot}3 \text{ ft.}
\end{aligned}$$

From observing now the distance the pile sinks from this stroke, another greater height may be calculated from which it would be safe to let the rammer fall. Should the pile, however, come upon a large stone, it might splinter when the rammer fell from a greater height than 5 feet.

46. A mass of W lbs. moving horizontally with a velocity of V feet per second is stopped by a chain (l feet long and a square inches effective sectional area), whose other end is

securely anchored. Find the greatest pull on the chain, E being the modulus of elasticity, and g the acceleration of a falling body.

$h = \frac{V^2}{2g}$ is the height through which W must fall from rest to acquire the velocity V. Hence kinetic energy of

$$W = W \text{ lbs.} \times h \text{ ft.}$$

$$= \frac{WV^2}{2g} \text{ ft.-lbs.}$$

This work must be done upon W to bring it to rest, hence it is the work W will do upon the chain.

Let $P =$ greatest pull on chain in lbs. at the instant W first comes to rest.

$$\text{stress} = \frac{P}{a} \text{ lbs. per sq. in.}$$

$$E = \frac{\text{stress}}{\text{strain}},$$

$$\text{or strain} = \frac{\text{stress}}{E} = \frac{P}{aE},$$

$$\text{elongation} = \text{strain} \times \text{length} = \frac{Pl}{aE},$$

$$\text{amt. of resilience} = \tfrac{1}{2} \text{ amt. of stress} \times \text{elongation}$$

$$= \frac{P}{2} \times \frac{Pl}{aE} = P^2 \cdot \frac{l}{2aE}.$$

Equating, $$P^2 \cdot \frac{l}{2aE} = \frac{WV^2}{2g}.$$

$$\therefore \quad P^2 = \frac{aEWV^2}{gl}.$$

$$\therefore \quad P = V\sqrt{\frac{aEW}{lg}}.$$

47. A waggon going down an incline is attached to a rope worked by a stationary engine. Its weight is five tons, and its velocity four miles per hour, when it is suddenly stopped by the accidental stopping or reversing of the engine. Find the greatest instantaneous pull upon the rope, which is 4 square inches in sectional area, there being 600 feet of free rope between waggon and engine. $E = 25{,}000{,}000$ lbs. and $g = 32$ feet per second per second.

$$\text{amount of res.} = \tfrac{1}{2}\text{ amount of stress} \times \text{elongation}$$

$$= \frac{P}{2} \times \frac{6P}{1{,}000{,}000} = \frac{3}{1{,}000{,}000} P^2 \text{ft.-lbs}$$

$$\text{kinetic energy of waggon} = 6023{\cdot}5 \text{ ft.-lbs.}$$

Equating, $P^2 = 2{,}007{,}833{,}333.$

$\therefore \quad P = 44{,}808$ lbs. $= 20$ tons.

48. A body of mass W lbs., moving with a velocity V feet per second, is connected by means of a chain l feet long, and of a square inches sectional area, to another body of mass w at rest. W and w being great compared to the mass of chain, its inertia is neglected. Find greatest pull upon chain.

As the chain begins to stretch W loses and w gains velocity, till both together with chain arrive for the first time at a common velocity, at which instant the tension is the greatest possible upon the chain, which will now begin to contract, causing w to move quicker than W, again extending as W increases till common velocity is arrived at; finally, after many such stretchings and contractings, the chain will settle at a fixed pitch of strain, and all will continue to move at common velocity above.

Let U = common velocity,
P = pull on chain in lbs. (at instant),
E = modulus of elasticity of chain,
WV = momentum when w was at rest,
$(W + w)\,U$ = momentum when at com. vel.

Equating, $$U = \frac{WV}{W + w}.$$

Owing to W's velocity being reduced, it has done a certain amount of work upon the chain, and since w has acquired velocity the chain has done a certain amount of work upon it, and the difference of these amounts of work equals the amount of resilience of chain.

From equation $h = \frac{v^2}{2g}$, where v is velocity acquired by a body falling from rest through h feet, we have

$$\frac{V^2}{2g} = \text{ht. } W \text{ must fall to acquire vel. } V,$$

and $$\frac{U^2}{2g} = \text{ht. } W \text{ must fall to acquire vel. } U.$$

$$\therefore \frac{V^2 - U^2}{2g} = \text{ht. } W \text{ must fall to increase from vel. } U \text{ to vel. } V.$$

and $$W . \frac{V^2 - U^2}{2g} = \text{work lost by } W.$$

$$w . \frac{U^2}{2g} = \text{work gained by } w.$$

$$\text{diff. of work} = \frac{1}{2g}\left\{ W(V^2 - U^2) - wU^2 \right\}.$$

$$= \frac{1}{2g}\left\{ WV^2 - (W + w)U^2 \right\}.$$

Putting for U its value,

$$= \frac{1}{2g}\left\{ WV^2 - (W + w)\left(\frac{WV}{W + w}\right)^2 \right\}$$

$$= \frac{1}{2g}\left(WV^2 - \frac{W^2V^2}{W + w} \right)$$

$$= \frac{V^2}{2g} \cdot \frac{Ww}{W + w} \text{ ft.-lbs.}$$

Now upon chain

$$\text{stress} \quad = \frac{P}{a} \text{ and } E = \frac{\text{stress}}{\text{strain}},$$

$$\text{or strain} \quad = \frac{\text{stress}}{E} = \frac{P}{aE},$$

$$\text{elongation} = \text{strain} \times \text{length} = \frac{Pl}{aE},$$

$$\text{amt. of res.} = \tfrac{1}{2} \text{ amt. of stress} \times \text{elongation.}$$

$$= \frac{P}{2} \text{lbs.} \times \frac{Pl}{aE} \text{ft.} = P^2 \frac{l}{2aE} \text{ ft.-lbs.}$$

Equating to difference of work,

$$P^2 . \frac{l}{2aE} = \frac{V^2}{2g} . \frac{Ww}{(W+w)}$$

$$P^2 = V^2 . \frac{aE}{lg} . \frac{Ww}{(W+w)}$$

$$\therefore \ P = V . \sqrt{\frac{aEWw}{lg(W+w)}} \text{ lbs.}$$

48*a*. Same problem by method of the Calculus. Let x_2 and x_1 be the spaces passed through by W and w respectively, and P the pull upon chain at any time t, hence at that instant

$$\text{elongation} = x_2 - x_1.$$

$$\text{strain} \quad = \frac{x_2 - x_1}{l} \qquad \text{def.}$$

$$\text{stress} \quad = \frac{P}{a}.$$

$$E = \frac{\text{stress}}{\text{strain}} = \frac{Pl}{a(x_2 - x_1)}.$$

$$\therefore P = \frac{Ea}{l}(x_2 - x_1) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since P accelerates w we have its acceleration

$$\frac{d^2 x_1}{dt^2} = \frac{P}{w} g = \frac{Eag}{wl} (x_2 - x_1),$$

and since P retards W its acceleration

$$\frac{d^2 x_2}{dt^2} = - \frac{P}{W} g = - \frac{Eag}{Wl} (x_2 - x_1),$$

subtracting,

$$\frac{d^2 (x_2 - x_1)}{dt^2} = - \frac{Eag}{l} \left(\frac{1}{W} + \frac{1}{w}\right) (x_2 - x_1).$$

A differential equation from which $(x_2 - x_1)$ is to be determined.

$$\text{Put } (x_2 - x_1) = c \cos (nt - e),$$

$$\text{then } \frac{d^2 (x_2 - x_1)}{dt^2} = - n^2 (x_2 - x_1),$$

Equating $$n^2 = \frac{Eag}{l} \left(\frac{1}{W} + \frac{1}{w}\right),$$

$$n = \sqrt{\frac{Eag\,(W + w)}{lWw}}.$$

When $$t = 0, \ x_2 - x_1 = 0.$$

$$\therefore 0 = c \cos (nt - e).$$

$$\therefore \cos (-e) = 0, \text{or } e = \frac{\pi}{2}.$$

$$\therefore (x_2 - x_1) = c \cos \left(nt - \frac{\pi}{2}\right)$$

$$= c \sin nt.$$

Differentiating,

$$\frac{dx_2}{dt} - \frac{dx_1}{dt} = nc \cos nt.$$

Since $\frac{dx_2}{dt}$ and $\frac{dx_1}{dt}$ are the velocities of W and w, their difference is zero at instant of greatest strain.

$$\therefore\ 0 = nc \cos nt.$$

$$\therefore\ nt = (2m + 1)\frac{\pi}{2},$$ where m is any integer.

$$\therefore\ t = \frac{2m+1}{2n}\pi,$$ gives times at which W and w come to a common velocity.

When $t = 0$, the difference of W and w's velocities is V.

$$\therefore \frac{dx_2}{dt} - \frac{dx_1}{dt} = V.$$

$$\begin{aligned} \therefore V &= nc \cos 0 \\ &= nc \\ \therefore\ c &= \frac{V}{n}. \end{aligned}$$

Sub. into

$$\begin{aligned} P &= \frac{Ea}{l}(x_2 - x_1) \\ &= \frac{Ea}{l} c \sin nt \\ &= \frac{Ea}{l}\frac{V}{n} \sin (2m+1)\frac{\pi}{2} \\ &= \frac{Ea}{l} \cdot \frac{V}{n} \\ &= \frac{Ea}{l} \cdot V \div \sqrt{\frac{Eag(W+w)}{lWw}} \\ &= V \cdot \frac{Ea}{l} \sqrt{\frac{lWw}{Eag(W+w)}} \\ &= V \cdot \sqrt{\frac{EaWw}{lg(W+w)}} \end{aligned}$$

Other things being constant, this is greatest when $W = w$, for $(W + w)$ constant.

Note also that the stress

$$p = \frac{P}{a} = V\sqrt{\frac{EWw}{alg\,(W + w)}},$$

which increases if a be decreased, although the amount of stress P decreases, so that the only way to lessen P without increasing $\frac{P}{a}$ is to increase l the length of chain.

49. A mass of 5 tons, intended to act as a drag upon a ship being launched, is connected by means of a chain 100 feet long and 8 square inches effective sectional area, and $E = 25{,}000{,}000$. Find the greatest pull on the chain if the ship, when floating, have inertia equivalent to a mass of 400 tons, and it be estimated that her velocity will be 20 feet per second when the drag comes into play.

$$\begin{aligned} P &= V\cdot\sqrt{\frac{aEWw}{lg\,(W + w)}} \\ &= 20\sqrt{\frac{8 \times 25000000 \times 896000 \times 11200}{100 \times 32 \times (896000 + 11200)}} \\ &= 20\sqrt{691358025} \\ &= 525874 \text{ lbs.} = 235 \text{ tons total pull.} \end{aligned}$$

$$\therefore p = \left(\frac{P}{a}\right) = \frac{235}{8} = 29 \text{ tons per square inch},$$

which is about the utmost stress that wrought iron can bear.

INTERNAL STRESS.

In the following pages, except specially stated, we premise that—

a. All forces and stresses are parallel to one plane.

b. That plane is the plane of the paper in all diagrams. Hence planes subjected to the stresses we are considering are shewn in diagrams by strong lines, their *traces*.

c. The diagrams represent slices of solid, of unit thickness normal to the paper; hence, the lengths of the strong lines are the areas of the planes.

d. The stresses are supposed constant both in direction and intensity which are normal to the paper, or every point on a diagram is in the same circumstances with respect to stress normal to the paper.

e. The relative position of two planes is measured by the angle between their normals.

f. The obliquity of the stress to the plane upon which it acts is the angle its direction makes with the normal to the plane.

These premises save a great deal of wording in the enunciations, and, therefore, of *apparent* complicity.

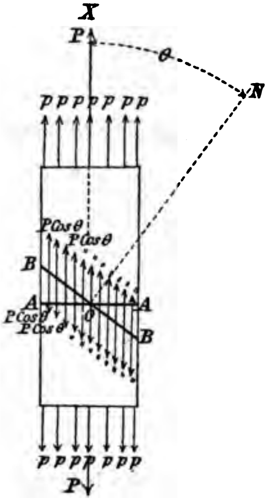


Fig. 9.

Internal stress at a point in a solid in a simple state of strain.

Let the axis OX be drawn in the direction of the stress P. Let AA be any section normal to this axis. Since the stress is uniformly distributed over AA, the intensity of the stress at all points of the plane AA is the same. Consider the point O, the intensity of the stress at O on the plane normal to OX is

$$p = \frac{\text{total stress}}{\text{area of plane}} = \frac{\text{P}}{\text{AA}}.$$

Through O draw any oblique plane, BB, whose normal, ON, makes the angle θ with OX. The stress on this plane is in the direction OX, and the amount of stress upon it is P (for the equilibrium of the parts). But the intensity of the stress on BB is less than p, since P is spread over a larger area than AA.

Since $$\text{AOB} = \theta,$$

and $$\frac{\text{OA}}{\text{OB}} = \cos \text{AOB},$$

$$\therefore \text{OB} = \frac{\text{OA}}{\cos \theta},$$

or $$\text{BB} = \frac{\text{AA}}{\cos \theta}.$$

$$\text{intensity of stress on BB} = \frac{\text{total stress}}{\text{area of plane}}$$

$$= \frac{\text{P}}{\text{BB}}$$

$$= \frac{\text{P}}{\left(\frac{\text{AA}}{\cos \theta}\right)}$$

$$= \frac{\text{P}}{\text{AA}} \,.\, \cos \theta$$

$$= p \,.\, \cos.\ \theta.$$

Hence the internal stress at all points within a solid, in a state of simple strain, is parallel to the direction of that stress—is greatest in intensity on the plane normal to that direction—on any other plane inclined at an angle θ to last, the intensity is one (cosine θ)th part of that intensity, *and zero on any* plane parallel to the direction of the stress.

The stress $p \cos \theta$ on BB being oblique to BB, it is convenient to resolve it into components normal and tangential to BB respectively.

The arrow $p \cos \theta$ represents the stress at the point O on the plane BB; from its extremity perpendiculars are dropped on ON and BB, which, by parallelogram of forces, give p_n and p_t, the intensities of the stresses upon BB, normal and tangential respectively.

Now $$\frac{p_n}{p \cos \theta} = \cos \theta \quad \text{def.}$$

$$\therefore \quad p_n = p \cos^2 \theta.$$

Also $$\frac{p_t}{p \cos \theta} = \sin \theta.$$

$$\therefore \quad p_t = p \sin \theta \cos \theta.$$

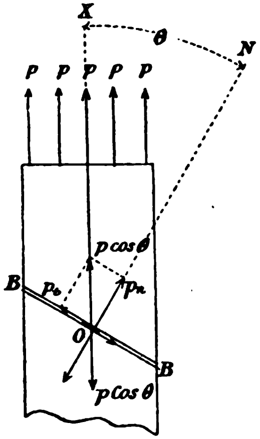


Fig. 10.

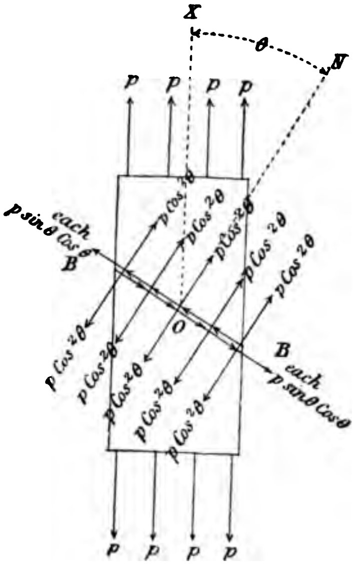


Fig. 11.

From the superposition of forces these two sets of forces may be considered independently of each other. For some cases in designing it might only be necessary to consider one set, if it were manifest that in providing for it there would be more than sufficient provision made for the other.

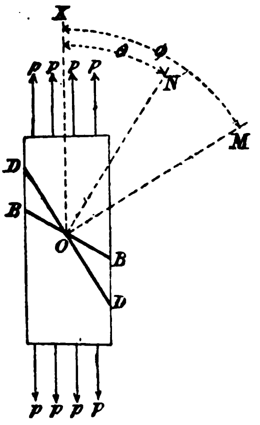


Fig. 12.

It is apparent from symmetry that for the plane CC inclined at the angle θ on the other side of the axis the stress is the same in all particulars as that on BB.

On a pair of planes whose obliquities are together equal to a right angle, the intensities of the tangential stresses are equal, and the sum of the intensities of the normal stresses equals the intensity of the initial stress.

Let BB be inclined at the angle θ, and DD at the angle ϕ,

where $$\theta + \phi = \frac{\pi}{2}.$$

On BB, $$\begin{cases} p_n = p \cos^2 \theta, \\ p_t = p \sin \theta \cos \theta. \end{cases}$$

On DD, $$\begin{cases} p_n{}' = p \cos^2 \phi, \\ p_t = p \sin \phi \cos \phi. \end{cases}$$

But $$\sin \theta = \cos \phi, \text{ and } \cos \theta = \sin \phi.$$

$\therefore$ $p_t = p_t{}'$, or the tangential component stresses have the same intensity on both planes.

Also $$\begin{aligned} p_n + p_n{}' &= p\,(\cos^2 \theta + \cos^2 \phi) \\ &= p\,(\cos^2 \theta + \sin^2 \theta) \\ &= p; \end{aligned}$$

or the sum of the intensities of the normal component stresses equals the intensity of the primary stress.

There are therefore at one point O four planes BB, DD, CC, *and EE, two* inclined on each side of OX, upon which the *tangential stress* has the same intensity.

Grouping together the pair of planes BB and EE, the one inclined at θ on the one side of OX, and the other at ϕ upon the opposite side, and therefore at $\theta + \phi$, or 90° to each other, we find that *at any point two planes being chosen at right angles to each other* the *tangential* or *shearing* stresses are of equal intensity, and the sum of the intensities of the *normal* stresses is equal to the intensity of the primary stress.

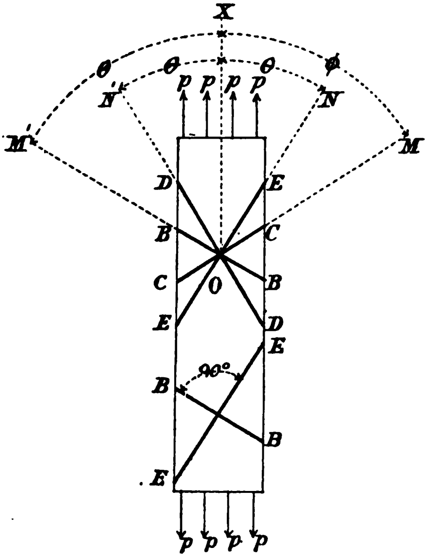


Fig. 13.

For all planes such as BB, DD, &c., that which is inclined at 45° sustains the tangential stress of greatest intensity, for

$$p_t = p \sin\theta \cos\theta,$$

$$= \frac{p}{2} \sin 2\theta.$$

$\therefore$ p_t is greatest when $\sin 2\theta$ is greatest.

" " $\sin 2\theta = 1.$

" " $2\theta = 90°$

" " $\theta = 45°$

The tangential stress on a plane such as BB is called a *shearing stress.* Many substances fracture under a shearing stress very readily. Notably cast iron under a strain of compression fractures by shearing along an oblique plane, the one portion sliding upon the other. The resistance then which cast iron offers to shearing is that which must be considered in designing short pillars to bear great loads. The planes upon which the intensity of the shearing stress is greatest, that is, planes inclined at 45° to the direct thrust, are those upon which it will shear. As the texture of the material is never homogeneous it may shear along planes more or less inclined than 45°, also the toughness of the skin will cause great irregularity.

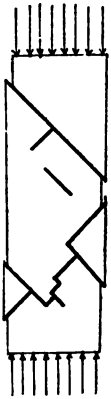


Fig. 14.

Brick stalks give way by the mortar shearing, and the upper portion sliding down an oblique section like a splice.

Examples.

50. A short pillar 2 square feet in area bears a load of 36 tons, find the intensity of the stress upon a plane section of it inclined at 20° to the axis, also the intensities of the normal and tangential component stresses on it.

$$p = \frac{80640}{288} = 280 \text{ lbs. per sq. inch}$$

on oblique plane.

$$\text{intensity} = p \cos\theta = 263{\cdot}1 \text{ lbs per sq. inch,}$$
$$p_n = p\cos^2\theta = 247{\cdot}3 \quad ,, \quad ,,$$
$$p_t = p \sin\theta \cos\theta = 90 \quad ,, \quad ,,$$

51. Find greatest intensity of shearing force on a plane *section of pillar* in last.

It will be upon a plane inclined at 45°.

$$p_t = p \sin 45^\circ \cos 45^\circ$$

$$= p \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{p}{2}$$

$$= 140 \text{ lbs. per sq. inch.}$$

52. A stone obelisk weighing 200 tons, and covering an area of 10 square feet, stands above a slate formation inclined to the horizontal 10°, find the intensity of the stress normal and tangential to the beds.

$$p_n = 301{\cdot}8 \text{ lbs. per sq. inch,}$$

$$p_t = 53{\cdot}2 \quad ,, \quad ,,$$

Compound State of Strain.

A solid is in *a* (*compound*) *state of strain* when subjected to two or more simple stresses in different directions simultaneously. We proceed to consider a solid in such a state of strain without enquiring how it was brought into that state; all its parts being supposed to be at rest, and all the parts into which it may be divided in equilibrium under the stresses exerted among each other, due to their elasticity, and those exerted at the external surface: but at the outset we do not regard those external stresses.

Upon any plane passing through a point within the solid, the stress at that point is definite in intensity and direction; for if along that plane the solid were divided into two parts, the mutual pressures between the cut surfaces at that point (no matter how complicated) can be compounded into one force, definite in amount and direction. Along this plane the intensity and direction of the stress varies, and at the point will only be constant over a very small part of the surface round it. If the stress be stated in lbs. per square inch, the total stress on this small surface which we are considering would be a mere fraction of the intensity. It will be convenient to *consider the* intensities of these stresses to be expressed, say, in lbs. per millionth part of a square inch, so that in the diagrams *two or three* arrows (*each* representing the *intensity*) may be drawn to represent the total amount of stress upon such small planes, without leading us to the supposition that they are of a few square inches in extent. And yet whatever results we arrive at are equally true for intensities expressed in the usual units, for the intensity at a point on a plane, upon which the intensity varies, can be expressed to any degree of accuracy in lbs. per square inch. Thus, at the point, the intensity of the stress in lbs. per square inch equals roughly, nearly, more nearly, &c.

Amt. of stress on the sq. in. surrounding point, roughly,
10 times amt. of stress on the $\frac{1}{10}$th of a sq. in. surrounding point.
100 times amt. of stress on the $\frac{1}{100}$th of a sq. in. surrounding point.
1,000,000 times amt. of stress on the $\frac{1}{1000000}$th of a sq. in. surrounding point, &c., &c.

Let OAO′B be a small rectangular parallelopiped at the point O in a solid in a state of strain.

Let q = intensity of stress on the faces OA and O′B at an obliquity a.
p = intensity of stress on faces OB and O′A at an obliquity β.

The normal components are—

$$p_n = p \cos \beta,$$
$$q_n = q \cos a.$$

The two sets of forces p_n directly balance each other, and may be removed, and also the two sets q_n, leaving the parallelopiped in equilibrium under the action of the tangential components.

$$p_t = p \sin \beta,$$
$$q_t = q \sin a.$$

The amount of tangential stress on OA and O′B is the intensity multiplied by area of face.

$$= q_t \,.\, \mathrm{OA}.$$

Also the amount on each of the faces OB and O′A.

$$= p_t . \text{OB}.$$

The two forces q_t.OA form a couple with a leverage OB tending to turn the parallelopiped in the direction in which the hands of a watch turn, while the two forces p_t . OB form a couple with a leverage OA tending to turn it in the opposite direction. Since the parallelopiped is in equilibrium under these two actions alone, the moments of these two couples must be equal.

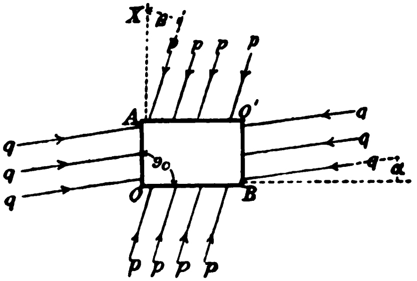
Force. Leverage.

$$q_t . \text{OA} \times \text{OB}$$
$$= p_t . \text{OB} \times \text{OA}.$$

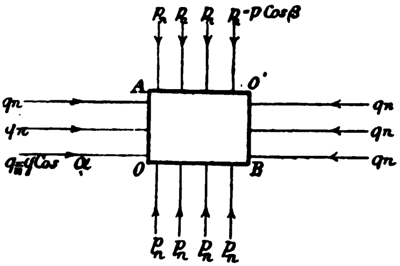
Now the area OA multiplied by the length OB gives the volume of the parallelopiped, and the area OB multiplied by the length OA also gives the volume.

$$\therefore\ q_t . V = p_t . V,$$
$$\therefore\ q_t = p_t.$$

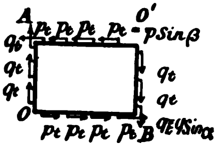
Hence, at a point within a solid in a state of strain, the tangential compon-



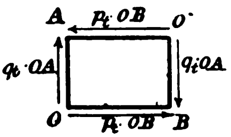
intensities and obliquities at O of stresses upon OA and OB



Normal components



Intensities of tangential components



Amounts of tangential stress

Fig. 15.

ents of the stresses upon any two planes through it at right angles to each other are of an equal intensity.

Cor.—If it be possible that the stress upon any plane through a point be wholly normal, then will the stress be wholly normal upon another plane at right angles to that plane.

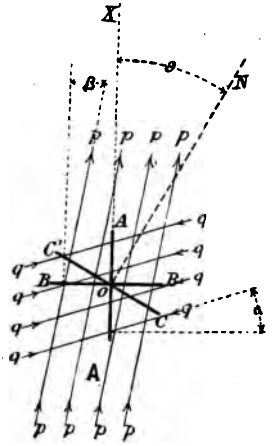


Fig. 16.

At a point within a solid in a state of strain the directions and intensities of the stresses upon two planes at right angles to each other being given, to find the direction and intensity of the stress upon any third plane.

Let q the intensity of the stress at O on the plane AA′ be inclined to the normal at the angle α;

Let p the intensity of the stress at O on the plane BB be inclined to the normal at the angle β;

These two planes being inclined to each other at a right angle. It is required to find the intensity and direction of the stress on a third plane CC′ through O inclined at any angle θ to AA.

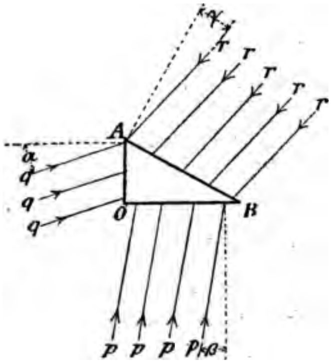


Fig. 17.

Consider a small triangular prism OAB at O bounded by portions of these three planes. It is in equilibrium under the stresses q on the face OA, p on the face OB, and the required stress r on the face AB. It is to be borne in mind that OAB is so small that all the planes OA, OB, and AB pass through the point O. Look upon OAB as a small part at O of the preceding diagram enlarged.

Choosing OX and OY along OA and OB as rectangular axes, and resolving p and q into components normal and tangential to the planes they act upon, we have

$$\begin{aligned} p_n &= p \cos \beta, \text{ normal to OB,} \\ q_n &= q \cos \alpha, \text{ normal to OA,} \\ \text{and } p_t &= p \sin \beta, \text{ tangential to OB,} \\ q_t &= q \sin \alpha, \text{ tangential to OA.} \end{aligned}$$

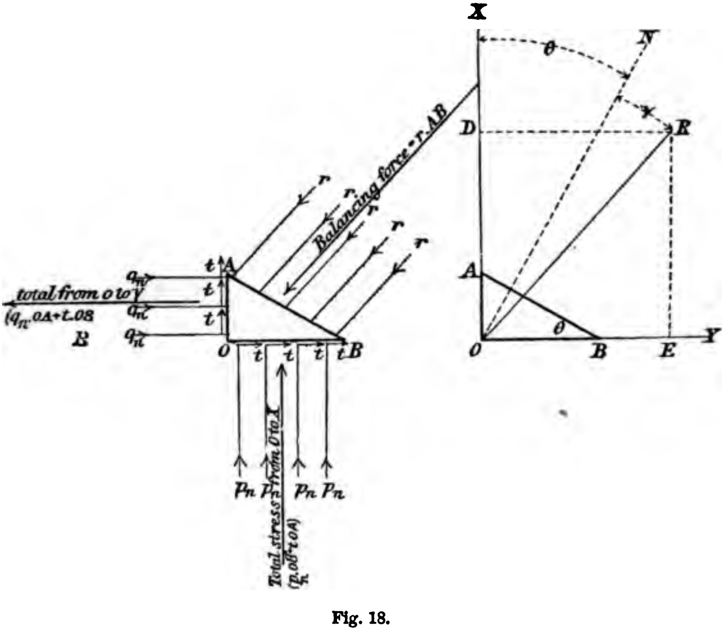


Fig. 18.

Since the intensities of the tangential component stresses upon the two planes OA and OB must be equal, one symbol t is put for both upon diagram

$$t = p_t = q_t = p \sin \beta = q \sin \alpha.$$

Note that $\text{OA} = \text{AB} \sin \theta$, and $\text{OB} = \text{AB} \cos \theta$.

Lay off OD = force parallel to the axis OX
= amount of normal stress on OB + amount of tangential stress on OA
$= p_n \,.\, \text{OB} + t \,.\, \text{OA}$
$= p_n \,.\, \text{AB} \cos\theta + t \,.\, \text{AB} \sin\theta$
$= \text{AB}\,(p_n \cos\theta + t \sin\theta).$

Lay off OE = force parallel to OY
= amount of normal stress on OA + amount of tangential stress on OB
$= q_n \,.\, \text{OA} + t \,.\, \text{OB}$
$= q_n \,.\, \text{AB} \sin\theta + t \,.\, \text{AB} \cos\theta$
$= \text{AB}\,(q_n \sin\theta + t \cos\theta).$

And completing the parallelogram we have for equilibrium

RO = amount of stress on AB in direction and magnitude,

since

$$\begin{aligned} \text{RO}^2 &= \text{OD}^2 + \text{OE}^2 \\ &= \text{AB}^2\{(p_n \cos\theta + t\sin\theta)^2 + (q_n \sin\theta + t\cos\theta)^2\} \\ &= \text{AB}^2\{p_n^2 \cos^2\theta + 2p_n t \sin\theta\cos\theta + t^2\sin^2\theta \\ &\qquad + q_n^2 \sin^2\theta + 2q_n t \sin\theta\cos\theta + t^2\cos^2\theta\}. \end{aligned}$$

Adding 1st and 4th terms, 2nd and 5th, 3rd and 6th within the brackets, and noticing in adding last pair that $\sin^2\theta + \cos^2\theta = 1$, we have

$$\text{RO}^2 = \text{AB}^2\{p_n^2\cos^2\theta + q_n^2\sin^2\theta + 2t\,(p_n + q_n)\sin\theta\cos\theta + t^2\}.$$

Now the intensity of the stress upon AB equals the total stress upon it divided by the area of the plane AB.

$$\therefore\ r = \frac{\text{RO}}{\text{AB}}$$

$$= \sqrt{\{p_n^2\cos^2\theta + q_n^2\sin^2\theta + 2t\,(p_n + q_n)\sin\theta\cos\theta + t^2\}};$$

also $$\tan \text{XOR} = \frac{\text{RD}}{\text{DO}}$$

$$= \frac{\text{OE}}{\text{OD}} = \frac{q_n \sin\theta + t\cos\theta}{p_n \cos\theta + t\sin\theta} \text{ gives XOR.}$$

Hence the obliquity of the stress r to the plane AB is

$$\gamma = \text{NOR}$$
$$= \text{XOR} - \theta.$$

Thus we have found r and v, the intensity and direction of the stress upon CC′ in terms of p, q, a and β. Hence, for a body in a possible state of strain, if at any point the stresses be given upon a pair of rectangular planes through it, the *internal stress* at that point is known. For the same point the internal stress might be known from having the stresses on different such pairs of rectangular planes. These are called *equivalent sets* of stresses. By a more elaborate process we might show that the internal stress at a point is known if the stresses on any pair of planes, not coincident, be given.

For some position of the plane CC′ the stress OR will coincide with the normal ON, and we will have

$$\theta = \text{XON} = \text{XOR}.$$

$$\therefore \quad \tan\theta = \frac{q_n \sin\theta + t\cos\theta}{p_n\cos\theta + t\sin\theta},$$

$$\text{or} \quad \frac{\sin\theta}{\cos\theta} = \frac{q_n\sin\theta + t\cos\theta}{p_n\cos\theta + t\sin\theta};$$

and clearing of fractions

$$p_n \sin\theta\cos\theta + t\sin^2\theta = q_n\sin\theta\cos\theta + t\cos^2\theta.$$

$$\therefore (p_n - q_n)\sin\theta\cos\theta = t(\cos^2\theta - \sin^2\theta).$$

$$\therefore \frac{p_n - q_n}{2} 2\sin\theta\cos\theta = t(\cos^2\theta - \sin^2\theta).$$

$$\therefore \frac{p_n - q_n}{2}\sin 2\theta \qquad = t\cos 2\theta.$$

$$\therefore \quad \frac{\sin 2\theta}{\cos 2\theta} = t \cdot \frac{2}{p_n - q_n}$$

$$\text{or} \quad \tan 2\theta = \frac{2t}{p_n - q_n}.$$

Two values of θ differing by a right angle will satisfy this equation; that is, there are two planes CC′ and FF′ inclined at an acute angle θ and an angle $90 + \theta$ respectively to AA′, and therefore at right angles to each other, upon which the stresses are wholly normal, the value of θ being such that the tangent of twice θ equals the ratio of sum of the common intensity of the tangential stresses upon AA′ and BB′ to the difference of the intensities of the normal stresses thereon.

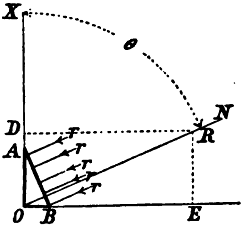


Fig. 19.

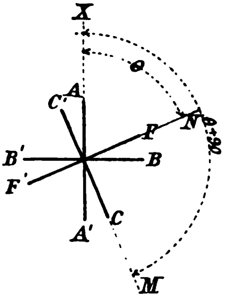


Fig. 20.

Now if we know the internal stress at O from having the stresses upon the pair of rectangular planes AA′ and BB′ given, we may calculate the stresses upon the pair of rectangular planes CC′ and FF′, and so express the internal stress at O by means of this equivalent set. Of all the equivalent sets at O this is the simplest by means of which to express the internal stress, as the intensities only, of the stresses upon CC′ and FF′ require to be specified, being normal, and the position specified of one only of the planes since they are rectangular.

Let AA′ and BB′ be the pair of rectangular planes through

O upon which the stresses are wholly normal, they are called the *planes of principal stress,* the stresses themselves the *principal stresses* at the point O, and the axes OX and OY the *axes of principal stress* at that point.

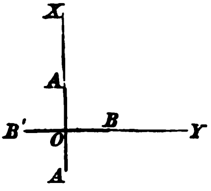
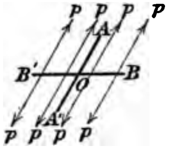


Fig. 21.

If the stress be oblique upon a plane through a point within a solid in a state of strain, and another plane be drawn through the point parallel to the stress thereon, then will the stress upon the second plane be parallel to the first plane.

At the point O let p be the stress on the plane BB′. Draw AA′ parallel to p, then will q be parallel to BB′.

Consider the equilibrium of the parallelopiped OAO′B at O.



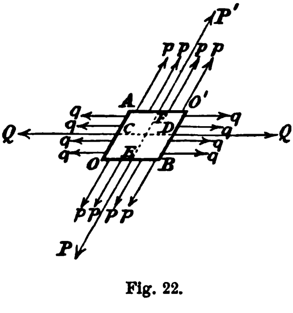


Fig. 22.

P and P' (the amount of the stress on the two faces OB and AO') are equal and in one straight line, being drawn parallel to OA through E and F, the middle points of those faces. Therefore they are in equilibrium, and may be removed, leaving Q and Q' in equilibrium themselves. Hence Q and Q' are in one straight line, and as this straight line passes through C and D, the middle points of the faces OA and BO', it is parallel to OB; that is, q is parallel to BB'.

These are called *a pair of conjugate stresses.* Being in equilibrium independently of each other, these stresses can be considered separately.

Cor. The principal stresses are also conjugate.

In the following examples the stresses are positive.

Examples.

53. At a point within a solid the stress on a plane BB′ through it is 90 lbs. per square inch, and inclined to the normal at an angle of 20°, while the normal component of the stress on another plane through it (AA′ at right angles to BB) is 60 lbs. per square inch. Find the total stress upon this other plane.

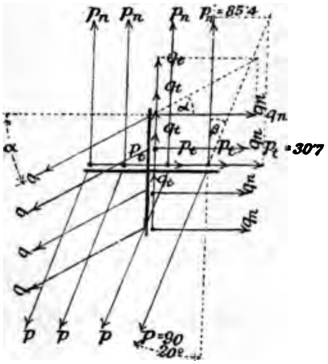


Fig. 23.

Given stress on BB′
$p = 90$ and $\beta = 20°$,

normal component
$p_n = 90 . \cos 20° = 84·41$,

tangent. component
$p_t = 90 . \sin 20° = 30·78$.

Let q be the required stress on AA′ and a its obliquity,

its normal component $q_n = 60$,

and $q_t = p_t = 30·78$.

Hence $q^2 = q_n^2 + q_t^2 = 4547·4$,

$\therefore$ $q = 67·4$ lbs. per sq. inch,

also $\cos a = \dfrac{q_n}{q} = ·89$.

$\therefore$ $a = 27° \, 7'$.

The total stress on AA′ is 67·4 lbs. per square inch, and is inclined at 27° 7′ to the normal.

54. AA′ and BB′ are a pair of rectangular planes through a point within a solid, the stress on BB′ is 3280 lbs. per square inch, and its obliquity is 10°. The normal component stress on AA′ being 2000 lbs. per square inch, find the intensity and obliquity of the total stress upon it.

q = *2080 lbs.* per sq. inch. $\alpha = 15° \, 57'$.

55. In last find the stress on CC′ a third plane through the point inclined at an angle of 30° to BB′. See diagram to general proposition, page 49, which is drawn to scale for this example.

$p_n = 3280 \cos 10°$, $q_n = 2000$, $t = 3280 \sin 10°$ or $2080 \sin 16°$.
$= 3230{\cdot}1.$ $= 569{\cdot}5.$

Lay off along OX the total amount of stress parallel to its direction.

$$\begin{aligned}
\text{OD} &= \text{amt. of normal stress on OB} + \text{amt. of tan. stress on OA}\\
&= p_n \,.\, \text{OB} + t \,.\, \text{OA}\\
&= p_n \,.\, \text{AB} \cos \theta + t \,.\, \text{AB} \sin \theta\\
&= \text{AB} \, (p_n \cos \theta + t \sin \theta)\\
&= \text{AB} \, (3230{\cdot}1 \times \cos 30° + 569{\cdot}5 \sin 30°)\\
&= \text{AB} \, (2797{\cdot}4 + 284{\cdot}8) = 3082{\cdot}2 \text{ AB}.
\end{aligned}$$

Lay off

$$\begin{aligned}
\text{OE} &= \text{total stress parallel to axis OY}\\
&= \text{amt. of nor. stress on OA} + \text{amt. of tan. stress on OB}\\
&= q_n \,.\, \text{OA} + t \,.\, \text{OB}\\
&= q_n \,.\, \text{AB} \sin \theta + t \,.\, \text{AB} \cos \theta\\
&= \text{AB} \, (q_n \sin \theta + t \cos \theta)\\
&= \text{AB} \, (2000 \sin 30° + 569{\cdot}5 \cos 30°)\\
&= \text{AB} \, (1000 + 404{\cdot}6) = 1404{\cdot}6 \text{ AB}.
\end{aligned}$$

now $\text{OR}^2 = \text{OD}^2 + \text{OE}^2 = 11472700 \text{ AB}^2.$

$\therefore$ $\text{OR} = 3387$ AB total stress on AB,

and $r = \dfrac{\text{total stress on AB}}{\text{area of AB}}$

$$= \frac{\text{OR}}{\text{AB}} = 3387 \text{ lbs. per sq. inch.}$$

Also $\tan \text{XOR} = \dfrac{\text{OE}}{\text{OD}} = \dfrac{1404{\cdot}6}{3082{\cdot}2} = {\cdot}4557.$

$\therefore$ $\text{XOR} = 24° \, 30'$ and $\gamma = \text{XOR} - 30 = -\, 5° \, 30'.$

The intensity of the stress on CC′ is 3,387 lbs. per square inch, and is inclined at an obliquity of 5° 30′.

56. In example 53 find the intensity and obliquity of the stress on a third plane through the point inclined to BB′ at 5°.

$$r = 93{\cdot}8 \text{ lbs. per sq. inch.} \quad \gamma = 17^\circ\ 28'.$$

57. The internal stress at a point within a solid in a state of strain being given 240 lbs. per square inch, of obliquity 8° on one plane BB′ through it, and 193 lbs. per square inch, of obliquity 10°, on the rectangular plane AA′. Find the position of the planes of principal stress.

$$t = \begin{cases} p_t = 240 \sin 8^\circ \ = 33{\cdot}4, \\ q_t = 193 \sin 10^\circ = 33{\cdot}4. \end{cases}$$

These must be equal or the stresses as given are impossible.

$$p_n = 240 \cos 8^\circ = 237{\cdot}6,$$
$$q_n = 193 \cos 10^\circ = 190.$$

Let θ be the inclination of the planes of principal stress to the plane BB′.

$$\tan 2\theta = \frac{2t}{p_n - q_n} = 1{\cdot}4034.$$
$$\therefore \quad 2\theta = 54^\circ\ 32' \text{ or } 234^\circ\ 32'$$
$$\therefore \quad \theta = 27^\circ\ 16' \text{ or } 117^\circ\ 16'.$$

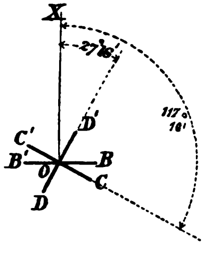


Fig. 24.

If BB′ be the plane upon which the given stress is 240, and OX the normal to it, then CC′ and DD′ are the planes of principal stress whose normals ON and OM make 27° 16′ and 117° 16′ with OX respectively.

58. Find the principal stress in example 57; that is, find the stress upon a third plane CC′, inclined at 27° 16′ to BB′.

$$\begin{aligned} \text{OD} &= p_n \,.\, \text{OB} + t \,.\, \text{OA} \\ &= \text{AB}\,(p_n \cos\theta + t \sin\theta) \\ &= \text{AB}\,(237{\cdot}6 \cos 27^\circ\ 16' + 33{\cdot}4 \sin 27^\circ\ 16') \\ &= 226{\cdot}5\ \text{AB}. \end{aligned}$$

$$\begin{aligned} \text{OE} &= q_n\,.\,\text{OA} + t\,.\,\text{OB} \\ &= \text{AB}\,(q_n \sin\theta + t\cos\theta) \\ &= 116{\cdot}7\ \text{AB}. \end{aligned}$$

$$\text{OR}^2 = \text{OD}^2 + \text{OE}^2 = 64910\ \text{AB}^2.$$

$$\text{OR} = 254{\cdot}8\ \text{AB}.$$

$$r = \frac{\text{OR}}{\text{AB}} = 254{\cdot}8 \text{ lbs. per sq. in,}$$

and is of course normal.

In finding the other principal stress—that is, on third plane DD′, inclined at 117° 16′ to BB′, we may use the functions of this angle and proceed as above, observing signs; but it is better to take the functions

of DOK = 62° 44′,
when OA = AB sin 62° 44′,
and OB = AB cos 62° 44′.

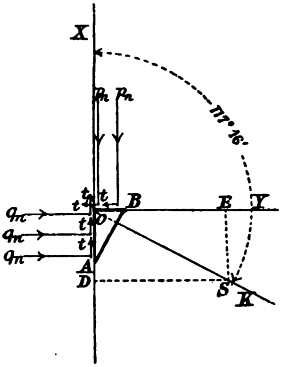


Fig. 25.

$$\begin{aligned} \text{OD} &= p_n\,.\,\text{OB} - t\,.\,\text{OA}, \text{ since } t \text{ acts against } p_n \\ &= \text{AB}\,(p_n \cos 62^\circ 44' - t \sin 62^\circ 44') \\ &= \text{AB}\,(108{\cdot}86 - 29{\cdot}69) \\ &= 79{\cdot}17\ \text{AB}. \end{aligned}$$

$$\begin{aligned} \text{OE} &= q_n\,.\,\text{OA} - t\,.\,\text{OB}, \text{ since } t \text{ acts against } q_n \\ &= \text{AB}\,(q_n \sin 62^\circ 44' - t\cos 62^\circ 44') \\ &= \text{AB}\,(168{\cdot}89 - 15{\cdot}3) \\ &= 153{\cdot}6\ \text{AB}. \end{aligned}$$

$$\text{SO}^2 = \text{OD}^2 + \text{OE}^2 = 29858\ \text{AB}^2.$$

$$\text{SO} = 172{\cdot}8\ \text{AB}$$

$$\text{S} = \frac{\text{SO}}{\text{AB}} = 172{\cdot}8.$$

The *principal* stresses are 254·8 and 172·8 lbs. per sq. inch.

59. In Ex. 57 find the stress on a plane GG′ inclined at 30° 39′ to BB′.

Directly from data as given in 57,

$$\begin{aligned}\text{OD} &= p_n \,.\, \text{OB} + t \,.\, \text{OA}\\ &= \text{AB}\,(p_n \cos\theta + t\sin\theta)\\ &= \text{AB}\,(237{\cdot}6 \cos 30^\circ\, 39' + 33{\cdot}4 \sin 30^\circ\, 39')\\ &= 221\ \text{AB}.\end{aligned}$$

$$\begin{aligned}\text{OE} &= q_n \,.\, \text{OA} + t \,.\, \text{OB}\\ &= \text{AB}\,(q_n \sin\theta + t\cos\theta)\\ &= \text{AB}\,(190 \sin 30^\circ\, 39' + 33{\cdot}4 \cos 30^\circ\, 39')\\ &= 125{\cdot}59\ \text{AB}.\end{aligned}$$

$$\text{OR}^2 = \text{OD}^2 + \text{OE}^2 = 64792\ \text{AB}^2.$$

$$\text{OR} = 254{\cdot}5\ \text{AB}.$$

$$r = \frac{\text{OR}}{\text{AB}} = 254{\cdot}5 \text{ lbs. per sq. in.}$$

Also $\tan \text{XOR} = \frac{\text{OE}}{\text{OD}} = {\cdot}5673.$

$$\therefore\ \text{XOR} = 29^\circ\, 34'$$

$$\begin{aligned}\gamma &= \theta - \text{XOR} = 30^\circ\, 39' - 29^\circ\, 34'\\ &= 1^\circ\, 5', \text{ obliquity of } r \text{ upon GG}'.\end{aligned}$$

To find the stress upon GG by finding the principal stresses, first as in 58, and then finding the stress upon GG from these.

The plane of greatest principal stress CC′ is inclined to BB′, the given plane, at 27° 16′; hence GG′ will be inclined to CC′ at 3° 23′. Hence we have the principal stresses 254·8 and 172·8 to find the stress upon GG′ inclined at 3° 23′.

Since there are no tangential stresses on CC′ and DD′ we have

$$\begin{aligned}\text{OD} &= p \,.\, \text{OB}\\ &= p \,.\, \text{AB} \cos\theta\\ &= \text{AB} \,.\, 254{\cdot}8 \cos 3^\circ\, 23'\\ &= 254{\cdot}3 \,.\, \text{AB}.\end{aligned}$$

$$
\begin{aligned}
\text{OE} &= q\,.\,\text{OA} \\
&= q\,.\,\text{AB}\sin\theta \\
&= \text{AB}\ 172{\cdot}8\sin 3^\circ\, 23' \\
&= 10{\cdot}2\,.\,\text{AB}. \\
\text{OR}^2 &= \text{OD}^2 + \text{OE}^2 = 64764\ \text{AB}^2. \\
\text{OR} &= 254{\cdot}5\ \text{AB}. \\
r &= \frac{\text{OR}}{\text{AB}} = 254{\cdot}5 \text{ lbs. per sq. in.} \\
\tan \text{XOR} &= \frac{\text{OE}}{\text{OD}} = {\cdot}04011. \\
\text{XOR} &= 2^\circ\, 18'. \\
\gamma &= \theta - \text{XOR} = 3^\circ\, 23' - 2^\circ\, 18' \\
&= 1^\circ\, 5', \text{ obliquity of } r \text{ on GG}'.
\end{aligned}
$$

60. At a point within a solid in a state of strain, the stresses upon a pair of rectangular planes through it are given—on BB′ the intensity of the normal component stress is 40, on AA′ the intensity of the normal component stress is 30, and the tangential component stresses are each of intensity 10. Find the obliquity to BB′ of the planes of principal stress, and find the principal stresses.

$$\text{Tan } 2\,\theta = 2. \quad \therefore \theta = 31^\circ 43' \text{ and } 121^\circ 43';$$

$$\text{for } \theta = \ \ 31^\circ\, 43' \begin{cases} \text{OD} = 39{\cdot}281\ \text{AB} \\ \text{OE} = 24{\cdot}277\ \text{AB} \end{cases} \therefore r = 46{\cdot}2,$$

$$\text{for } \theta = 121^\circ\, 43' \begin{cases} \text{OD} = 12{\cdot}522\ \text{AB} \\ \text{OE} = 20{\cdot}26\ \text{AB} \end{cases} \therefore s = 23{\cdot}8.$$

61. In 60 find the stress on a plane inclined at 46° 43′ to BB′. Deducting 31° 43′, we find the plane to be inclined at 15° to the plane of greatest principal stress.

Using results of 60, and t being then zero,

$$r = 45 \text{ and } \gamma = 7^\circ\, 8'.$$

RANKINE'S METHOD OF ELLIPSE OF STRESS.

The preceding method of finding the stress upon any particular plane through a point at which the state of strain is known, is too tedious to be readily remembered or applied, and becomes intricate when the stresses are some thrusts and others tensions. We now proceed to a general method. Having already proved that there is a pair of principal stresses at a point, we proceed, upon the supposition that these are given, to find the stress on a third plane through the point.

Equal-like principal stresses. If the pair of principal stresses at a point be like (both thrusts or both tensions), and be of equal intensity, the stress on any third plane through the point is of that same intensity, and is normal to the plane.

Let AA′ and BB′ be the planes of principal stress at the point O, and let the intensities of the principal stresses, p and q, be equal and alike (both thrusts).

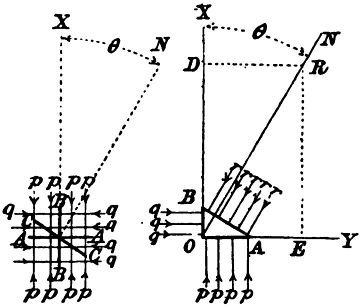


Fig. 26.

CC′ is any third plane through O, inclined at θ to AA′.

OAB is a small triangular prism at O, having its faces in those planes. This prism is in equilibrium under the three forces—the total thrusts under OA, OB, and AB.

Lay off OD = total stress parallel to OY

= p . OA,

and OE = total stress parallel to OX

= q . OB.

Complete the parallelogram.

Then RO represents the total stress on AB in direction and amount.

$$\tan \mathrm{ROD} = \frac{\mathrm{OE}}{\mathrm{OD}} = \frac{q\,.\,\mathrm{OB}}{p\,.\,\mathrm{OA}} = \frac{\mathrm{OB}}{\mathrm{OA}}, \text{ since } p = q.$$

$$\therefore \mathrm{ROD} = \mathrm{OAB}$$
$$= \theta. \quad \therefore \mathrm{OR} \text{ is upon } \mathrm{ON}.$$

Hence RO is normal to AB.

And
$$\mathrm{RO}^2 = \mathrm{OD}^2 + \mathrm{OE}^2$$
$$= p^2\,.\,\mathrm{OA}^2 + q^2\,.\,\mathrm{OB}^2$$
$$= p^2\,(\mathrm{OA}^2 + \mathrm{OB}^2), \text{ as } p = q,$$
$$= p^2\,.\,\mathrm{AB}^2.$$
$$\therefore \mathrm{RO} = p\,.\,\mathrm{AB}.$$

Now
$$r = \frac{\text{amount of stress on AB}}{\text{area of AB}}$$
$$= \frac{\mathrm{RO}}{\mathrm{AB}}$$
$$= p \text{ or } q.$$

COR. Every plane through O is a plane of principal stress. Each point in a fluid is in this state of strain.

Equal-unlike principal stresses.—If the pair of principal stresses at a point be unlike (one a thrust and the other a tension), and be of equal intensity, the stress on any third plane through the point is of that same intensity, and is inclined at an angle to the normal to the plane of principal stress, equal to that which the normal to this third plane makes therewith, but upon the *opposite side.*

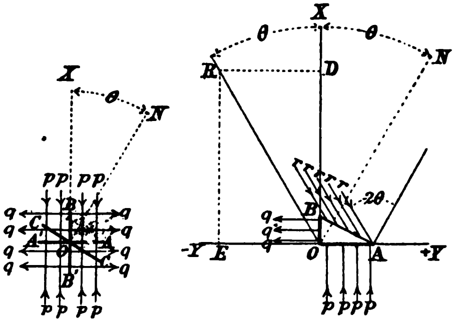


Fig. 27.

Let AA′ and BB′ be the planes of principal stress at the point O; p and q the intensities of the principal stresses of equal value, p being a thrust and q a tension; and CC′ any third plane through O inclined at θ to AA′.

OAB, a small triangular prism at O bounded by these three planes, is in equilibrium under the three forces, viz., the amounts of stress on its faces OA, OB, and AB.

Lay off OD = total stress parallel to OX,

$= p \,.\, \text{OA}$,

and OE = total stress parallel to OY in the direction of q,

$= q \,.\, \text{OB}$.

Complete the parallelogram ODRE. Then RO represents the total stress on AB in direction and amount.

$$\therefore \quad \tan \text{ROD} = \frac{\text{OE}}{\text{OD}} = \frac{q \,.\, \text{OB}}{p \,.\, \text{OA}} = \frac{\text{OB}}{\text{OA}} = \tan \text{OAB}$$

$$\therefore \quad \text{ROD} = \text{OAB} = \theta.$$

That is, RO is inclined at the same angle to the axis OX as ON is, but on the opposite side. Hence the inclination of RO to the normal ON is $2\,\theta$.

Again

$$\begin{aligned} \text{RO}^2 &= \text{OD}^2 + \text{OE}^2 \\ &= p^2 \,.\, \text{OA}^2 + q^2 \,.\, \text{OB}^2 \\ &= p^2 (\text{OA}^2 + \text{OB}^2) \\ &= p^2 \,.\, \text{AB}^2; \\ \therefore \quad \text{RO} &= p \,.\, \text{AB}, \\ r &= \frac{\text{amount of stress on AB}}{\text{area of AB}} \\ &= \frac{\text{RO}}{\text{AB}} \\ &= p \text{ or } q. \end{aligned}$$

Consider the triangle of forces OER, we have OE drawn from O in the direction of q, then ER drawn from E in the direction of p; hence RO, taken in the same order, is the direction of r.

If θ be greater than 45°, r is like q.

If θ equals 45°, r is entirely tangential to AB.

If θ be less than 45°, r is like p.

Hence, if the principal stresses at a point be equal and unlike the stress on a third plane, is of that same intensity, is like the stress on the plane it is least inclined to, and its direction is inclined to the axis at the same angle as the normal is, but upon the opposite side. If the new plane be inclined at 45°, the stress is entirely tangential.

The principal stresses at a point within a solid in a state of strain being given, to find the intensity and obliquity of the stress at that point on a third plane through it.

AA′ and BB′ are the planes of principal stress at O; p and q are the principal stresses. Let p be the greater, and let them be both positive, say both tensions. It is required to find r, the intensity of the stress upon CC′, and γ, the angle it makes with ON, the normal to CC′. θ is the inclination of CC′ to AA′, the plane of greatest principal stress.

Of two unequal quantities the greater is equal to the sum of their *half sum* and their *half difference*, while the lesser equals their difference.

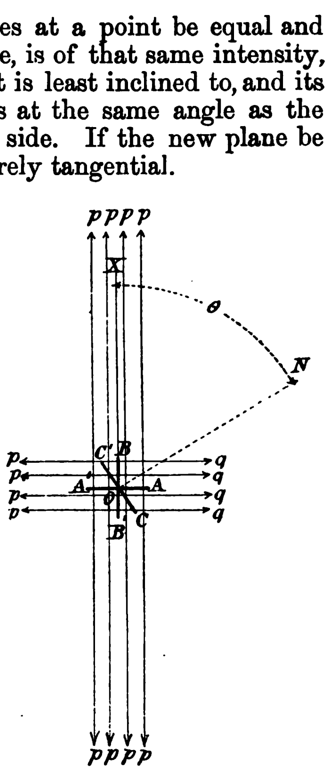


Fig. 28.

$$\therefore\quad p = \frac{p+q}{2} + \frac{p-q}{2}, \text{ an identity,}$$

$$\text{and } q = \frac{p+q}{2} - \frac{p-q}{2}, \qquad ,,$$

We may look upon the plane AA′ as bearing two separate tensions of intensities $\frac{p+q}{2}$ and $\frac{p-q}{2}$ in lieu of the tension of intensity p; and on the plane BB′ as bearing a tension of intensity $\frac{p+q}{2}$ and a thrust of intensity $\frac{p-q}{2}$ in lieu of the tension of intensity q. We may now group these together in pairs, thus: the tension on AA′ of intensity $\frac{p+q}{2}$ along with the tension on BB′ of intensity $\frac{p+q}{2}$, and the tension on AA′ of intensity $\frac{p-q}{2}$ along with the thrust on BB′ of intensity $\frac{p-q}{2}$. Then find separately for each pair the stress upon CC′, and finally compound these two stresses on CC′ by means of the triangle of forces. The first pair is a pair of *equal-like* principal stresses (both tensions of intensity $\frac{p+q}{2}$). So the consequent stress on CC′ will be a tension of intensity $\frac{p+q}{2}$, and normal to CC′. The second pair is a pair of *equal-unlike* principal stresses *of intensity* $\frac{p-q}{2}$ (a tension on AA′ and a thrust on BB′),

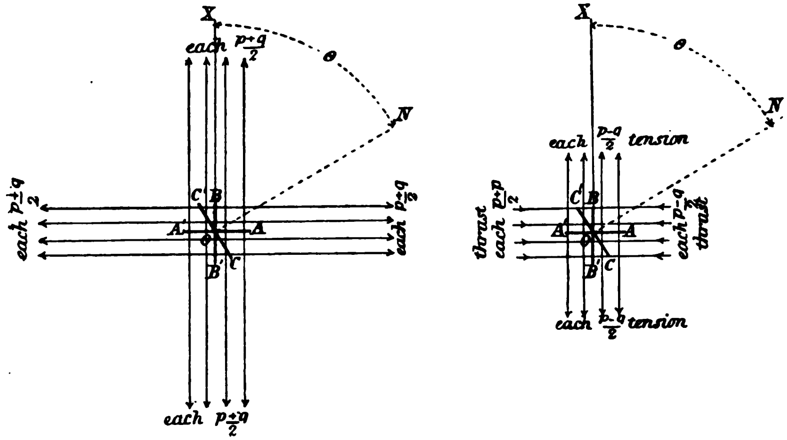

X
each $\frac{p+q}{2}$
θ
N
each $\frac{p+q}{2}$
each $\frac{p+q}{2}$
C
B
A
A
O
B′
C′
each $\frac{p+q}{2}$
X
θ
N
each $\frac{p-q}{2}$ tension
thrust each $\frac{p+p}{2}$
each $\frac{p-q}{2}$ thrust
C
B
A
A
O
B′
C
each $\frac{p-q}{2}$ tension

Fig. 29.

so the consequent stress on CC′ will be of intensity $\frac{p-q}{2}$ and inclined at an angle θ upon the side of OX opposite from that upon which ON is.

Figure shows these partial resultant stresses on AB, a very small part of CC′, at O. To find the total resultant stress upon CC′, it remains to compound these by the triangle of forces. From O lay off OM $= \frac{p+q}{2} =$ the intensity of the first partial stress and in the direction thereof, *i.e.*, along ON. From M draw MR $= \frac{p-q}{2} =$ the intensity of the second partial stress and in the direction thereof, *i.e.* parallel to OS, which direction is most conveniently found by describing from M as centre with radius MO a semicircle QOP, and joining QMP.

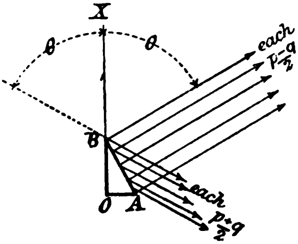


Fig. 30.

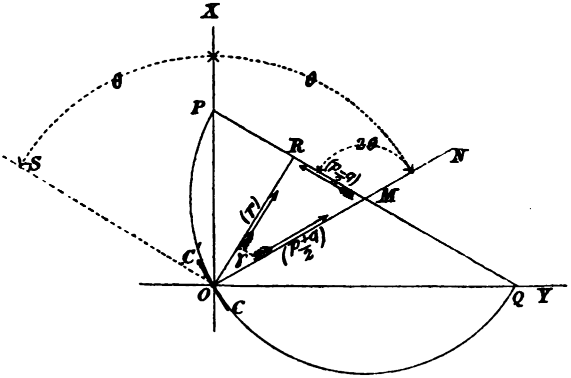


Fig. 31.

Then will OR, the third side of the triangle OMR, taken in the *opposite* order (see arrows) be the direction and intensity of the resultant stress r on CC′.

The preceding construction, as shown on last figure, is geometrically all that is required, p and q being given to find r; the text and figures given before being the development and proof.

From the construction note that

$$\text{MP} = \text{MQ} = \text{OM} = \frac{p+q}{2};$$

$$\begin{aligned} \text{also} \quad \text{QR} &= \text{MQ} + \text{MR} \\ &= \frac{p+q}{2} + \frac{p-q}{2} \\ &= \text{P}, \\ \text{and} \quad \text{PR} &= \text{MP} - \text{MR} \\ &= \frac{p+q}{2} - \frac{p-q}{2} \\ &= q; \\ \text{RMN} &= 2\theta, \\ \text{ROM} &= \gamma, \text{ the obliquity of } r. \end{aligned}$$

Normal and tangential components of r, the stress on the third plane CC′.

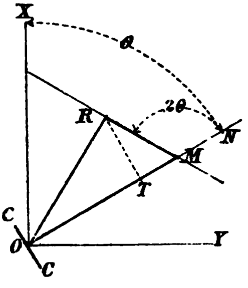


Fig. 32.

Drop RT perpendicular to ON. The tangential component of r is

$$\begin{aligned} r_t &= \text{RT} \\ &= \text{MR} \sin \text{RMT} \\ &= \frac{p-q}{2} \sin 2\theta \\ &= (p-q) \sin \theta \cos \theta, \end{aligned}$$

since $\sin 2\theta$

$$\begin{aligned} &= \sin (\theta + \theta) \\ &= \sin \theta \cos \theta + \cos \theta \sin \theta \\ &= 2 \sin \theta \cos \theta. \end{aligned}$$

COR.—If DD′ be the plane at right angles to CC′, its inclination to the axis OX being $\theta' = (\theta + 90°)$, the sine of which *equals* $\cos \theta$ and the cosine of which equals $-\sin \theta$; the *value of* r_t *for* DD′ will be the same as above, that is, the tangential components of the stresses on any pair of rectangular planes is the same. This we arrived at by the general method, page 42, which compare.

The normal component of r is

$$\begin{aligned}
r_n &= \text{OT} \\
&= \text{OM} - \text{MT} \\
&= \text{OM} - \text{MR} \cos \text{RMT} \\
&= \text{OM} + \text{MR} \cos \text{RMN}
\end{aligned}$$

(These two angles have the same cosine, but of opposite sign.)

$$\begin{aligned}
&= \text{OM} + \text{MR} \cos 2\theta \\
&= \text{OM} \times 1 + \text{MR} . \cos (\theta + \theta) \\
&= \frac{p+q}{2} (\cos^2\theta + \sin^2\theta) + \frac{p-q}{2} (\cos^2\theta - \sin^2\theta) \\
&= \cos^2\theta \left(\frac{p+q}{2} + \frac{p-q}{2}\right) + \sin^2\theta \left(\frac{p+q}{2} - \frac{p-q}{2}\right) \\
&= p . \cos^2\theta + q . \sin^2\theta.
\end{aligned}$$

COR.—If s_n be the normal component of stress on DD′ the plane at right angles to CC′, whose inclination to OX is $\theta' = (\theta + 90°)$, then

$$s_n = p \cos^2 \theta' + q \sin^2 \theta'.$$

But $\qquad \cos \theta' = - \sin \theta$, and $\sin \theta' = \cos \theta$.

$$\therefore \quad s_n = p \sin^2 \theta + q \cos^2 \theta.$$

Now, $\qquad r_n = p \cos^2 \theta + q \sin^2 \theta,$

and adding, we get

$$\begin{aligned}
s_n + r_n &= p (\sin^2\theta + \cos^2\theta) + q (\sin^2\theta + \cos^2\theta) \\
&= p + q.
\end{aligned}$$

That is, the sum of the normal components of the stresses on any pair of rectangular planes is equal to the sum of the principal stresses. From these two corollaries verify Ex. 58, where principal stresses are found; and Ex. 57, where the normal components on a pair of rectangular planes are found. Verify Ex. 60, which could have been solved by these two corollaries arithmetically.

As CC′ moves through all positions, M moves in a circle round O, and R moves in a circle round M, OM and MR keeping equally inclined to the vertical on opposite sides of it.

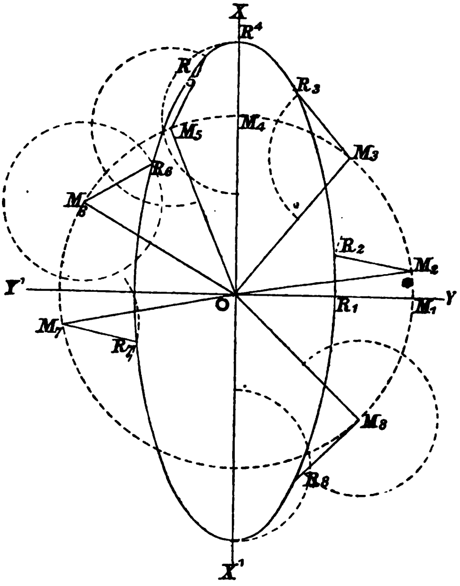


Fig. 33.

The diagram shows their positions for eight positions of the plane CC′. The locus of R is an ellipse, the major semi-axis being

$$\begin{aligned} OR_4 &= OM_4 + M_4R_4 \\ &= \frac{p+q}{2} + \frac{p-q}{2} \\ &= p; \end{aligned}$$

and the minor semi-axis is

$$\begin{aligned} OR_1 &= OM_1 - M_1R_1 \\ &= \frac{p+q}{2} - \frac{p-q}{2} \\ &= q. \end{aligned}$$

This is called *the ellipse of stress* for the point O within a solid in a state of strain. Its principal axes are the normals to the planes of principal stress, the principal semi-axes being equal to the intensities of the principal stresses. The radius-vectors OR_2, OR_3, &c., are the stresses in direction and intensity upon the planes at O to which OM_2, OM_3, &c., are respectively the normals.

The ordinary trammel for constructing ellipses consists of a piece like PRQ, whose extremities P and Q slide in two grooves, XOX and YOY, at right angles to each other, while the point R traces an ellipse whose semi-axes are PR and QR.

When Q arrives at O, R is at A and OA = QR = P; when P arrives at O, R is at B and OB = PR = q (page 66).

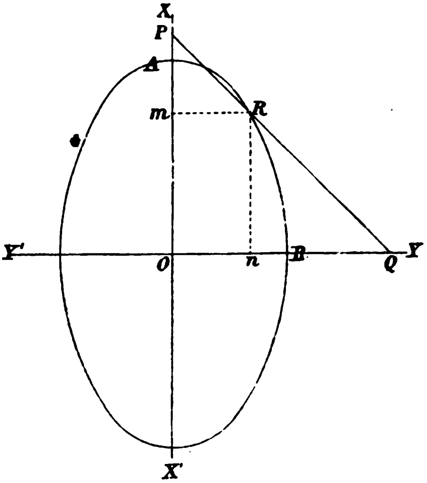


Fig. 34.

Taking O as origin, the co-ordinates of R are

$$x = om;\ y = on.$$

$$\therefore \quad x = n\mathrm{R} = \mathrm{QR} \,.\, \cos\theta = p \,.\, \cos\theta;$$

$$\text{and } y = m\mathrm{R} = \mathrm{PR}\sin\theta = q\sin\theta.$$

$$\therefore \quad \frac{x}{p} = \cos\theta, \text{ and } \frac{y}{q} = \sin\theta.$$

$$\therefore \frac{x^2}{p^2} + \frac{y^2}{q^2} = \cos^2\theta + \sin^2\theta$$

$$= 1,$$

the ordinary equation to an ellipse in terms of the semi axes p and q.

If p and q *are* both thrusts, it is convenient to consider a thrust positive, and the proof is exactly the same, all the sides of OMR representing the opposite kind of stress from what they did in last case.

When p and q are unlike, the kind of stress of which the greater p consists, is to be considered positive.

Thus, if $p > q$ and p, a tension while q is a thrust. The preceding proof will hold if q be considered to include its negative sign; but in this case if $(-q)$ be substituted for q, we have

$$\text{OM} = \frac{p-q}{2}, \text{ and MR} = \frac{p+q}{2}.$$

Hence, the proposition is proved generally.

It is important to notice that OM and MR are both always positive, that is like p the greater principal stress, and that

$$\text{OM} > \text{MR, if } q \text{ is positive (like } p\text{),}$$

$$\text{and OM} < \text{MR, if } q \text{ is negative (unlike } p\text{).}$$

An advantage of this geometrical method, the ellipse of stress, is that we are now in a position to examine the value and sign of r, the stress upon a third plane CC′, and of its normal and tangential components for special positions of that plane. OM is always normal to CC′, while MR generally is resolvable into two components, one tangential to CC′ and the other normal, which last has to be either added to, or subtracted from, OM to give the total normal component according as OMR is an obtuse or an acute angle.

a. *Positions of* CC′ *for which* r *the stress upon it will have the greatest or least value.*

Since OM and MR are constant, OR increases as the angle OMR increases, is greatest when OMR = 180° and OM and MR are in one straight line and a continuation one of the other when

$$\text{OR} = \text{OM} + \text{MR},$$

$$\text{or } r = \frac{p+q}{2} + \frac{p-q}{2} = p;$$

and OR is least when ∠ OMR is zero, and OM and MR again in one straight line, but MR lapping back on OM, when

$$\text{OR} = \text{OM} - \text{MR},$$

$$\text{or } r = \frac{p+q}{2} - \frac{p-q}{2} = q.$$

Hence the planes of principal stress are themselves the planes of greatest and least stress.

b. *Position of* CC′ *for which the intensity of the shearing stress has the greatest value.*

As OM is always normal to CC′ it does not give any tangential component, whereas MR assumes all positions as CC′ changes, and will give a component tangential to CC′, which will be the greatest possible when MR is altogether tangential to CC′. Hence the position of CC′, which makes MR parallel to CC′, is that for which the shearing stress has the greatest possible intensity.

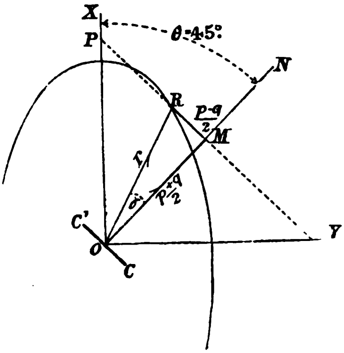


Fig. 35.

Hence intensity of shearing stress $= \text{MR} = \frac{p-q}{2}$.

And since MR is parallel to CC′ and OM normal to it,

$$\therefore\ \text{OMR} = 90^\circ,$$

and triangle MOP being isosceles, we have

$$\begin{aligned}\theta &= \text{inclination of CC}' \\ &= \text{MOP} \\ &= 45^\circ.\end{aligned}$$

And we know that the tangential stress is the same on the section perpendicular to CC′. That is, the planes of greatest tangential stress are the two planes inclined at 45° to the axes.

We saw (page 44) that cast iron subjected to a simple thrust would give way by shearing along the plane inclined at 45° to the thrust. We now infer that if it be in a compound state of strain it will most readily give way by shearing along a plane inclined at 45° to the planes of principal stress.

c. *Position of* CC′ *for which the total stress* R *upon it will be entirely tangential.*

When q is like p, it is impossible for the stress to be entirely tangential to CC′, because OM > MR, and however acute OMR may be, the normal component of MR, which has to be subtracted from OM to give the total normal stress, cannot be greater than MR itself, and consequently is always less than OM, and so there will always be a remainder; that is, for all positions of CC′ there is a normal component stress, and the total stress can never be entirely tangential.

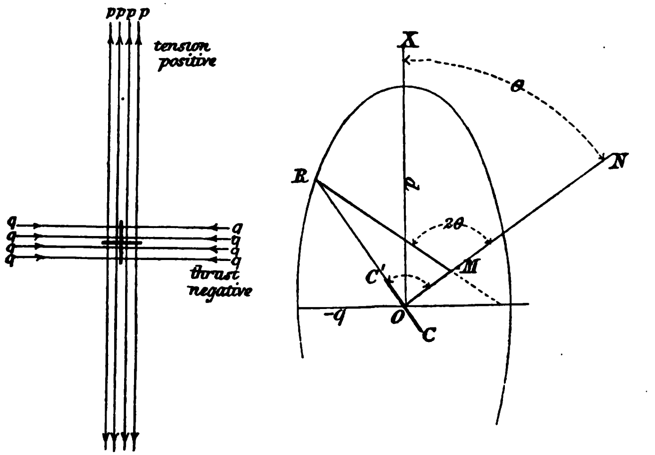


Fig. 36.

But when q is unlike p, then OM < MR, and for the particular position of CC′, when the angle OMR is of such an acuteness that the normal component of MR, which has to *be subtracted* from OM to give the total normal stress, is exactly of the same length as OM; then the total normal stress will be zero, and the total stress r entirely tangential.

This occurs when R is in one straight line with CC′. ROM is then a right angle, making MO the normal component of MR, equal and opposite to OM, which it destroys, leaving the total stress OR tangential to CC′; its magnitude is found thus:

$$\text{OR}^2 = \text{MR}^2 - \text{OM}^2 \text{ (Euc. I. 47).}$$

$$\text{or} \quad r^2 = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2$$

$$= p \cdot q,$$

$$\therefore \quad r = \sqrt{pq},$$

i.e., the stress on CC′ is the geometrical mean of the principal stresses. Also

$$2\theta = \text{RMN},$$

$$\therefore \quad \cos 2\theta = \cos \text{RMN}$$

$$= -\cos \text{RMO}$$

$$= -\frac{\text{MO}}{\text{MR}}$$

$$= -\frac{p-q}{p+q},$$

which determines θ, the position of CC′ for which the total stress is tangential.

Here we must guard against supposing that the above is the position of CC′ for which the tangential stress has the greatest intensity, for case (*b*) holds for all conditions of p and q; that is, the tangential stress on CC′ when inclined at 45°, although only a component of the total stress, will be of greater intensity than the total tangential stress in case (*c*).

d. *Position of* CC′ *for which* γ*, the obliquity of the stress thereon, is the greatest possible.*

When q is unlike p, case (*c*) is the solution, for in it $\gamma = \text{RON} = 90°$, the greatest possible.

When q is like p, OM > MR; and the obliquity of OR, the stress on CC′ is greatest when γ = ROM is the greatest possible of all triangles constructed with OM and MR for two of their sides. This occurs when ORM is a right angle.

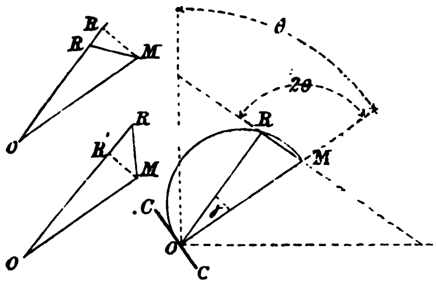


Fig. 37.

For suppose the triangle OMR constructed with ORM not a right angle; then drop MR′ at right angles to OR. It is evident that MR′ is less than MR. Now sin ROM $= \frac{\mathrm{MR}'}{\mathrm{MO}}$ is greatest when MR′ is greatest; that is, when MR′ = MR; that is, when ORM is a right angle, and ROM is greatest when its sine is greatest.

In this case the intensity of the stress is

$$\mathrm{OR}^2 = \mathrm{OM}^2 - \mathrm{MR}^2,$$

$$\text{or} \quad r^2 = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2$$

$$= pq,$$

$$r = \sqrt{pq},$$

a geometrical mean between the principal stresses.

$$\text{Also} \quad 2\theta = \mathrm{RMN},$$

$$\cos 2\theta = \cos \mathrm{RMN}$$

$$= - \cos \mathrm{ROM} = - \frac{\mathrm{MR}}{\mathrm{OM}}$$

$$= - \frac{p-q}{p+q},$$

which determines θ, the position of CC′ for which the stress has the greatest obliquity possible.

Note that these values of r and $\cos 2\theta$ are the same as those of (*c*), and that whether p and q are like or unlike. But this is not the case with γ, the obliquity, which is 90° when p and q are unlike, and has $\frac{p-q}{p+q}$ for its sine when p and q are alike.

Examples.

62. At a point within a solid in a state of strain the principal stresses are tensions of 255 lbs. and 171 lbs. per square inch. Find the stress on a plane inclined at 27° to the plane of greatest principal stress (converse of Ex. 58).

Data. $p = 255$, $q = 171$, and $\theta = 27°$; hence $\frac{p+q}{2} = 213$ and $\frac{p-q}{2} = 42$.

Construction. OX and OY, the axes of principal stresses, draw ON the normal to CC′, making XON$=\theta=27°$. Lay off along it OM $= \frac{p+q}{2} = 213$. From M as centre with radius MO, describe semicircle POQ and join PMQ; lay off from M towards P, MR $= \frac{p-q}{2} = 42$. This construction makes MR to be inclined to OX at an angle $\theta = 27°$, but upon the opposite side of it from OM.

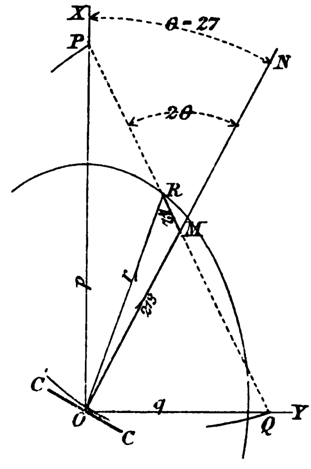


Fig. 38.

Looking upon the principal stresses as a pair of like principal stresses, tensions of intensities 213, together with a pair of unlike principal stresses, a tension and a thrust of intensities 42. Then OM represents a tension 213 upon plane CC′ due to first group, and MR the tension 42 upon CC′ *due to second* group; hence OR, the third side of the triangle, taken in the opposite direction, represents the total stress upon CC′ in direction and intensity.

$$\mathrm{OR}^2 = \mathrm{OM}^2 + \mathrm{MR}^2 - 2\mathrm{OM} \,.\, \mathrm{MR} \cos \mathrm{OMR},$$

$$\text{but } \cos \mathrm{OMR} = - \cos \mathrm{RMN} = - \cos 2\theta.$$

$$\therefore \quad \mathrm{OR}^2 = \mathrm{OM}^2 + \mathrm{MR}^2 + 2\mathrm{OM} \,.\, \mathrm{MR} \cos 2\theta.$$

$$\therefore \quad r^2 = 45369 + 1764 + 17892 \cos 54^\circ$$

$$= 57649.$$

$$\therefore \quad r = 240 \text{ lbs.}$$

$$\text{Also } \frac{\sin \gamma}{\sin 2\theta} = \frac{\sin \mathrm{ROM}}{\sin \mathrm{OMR}} = \frac{\mathrm{MR}}{r}.$$

$$\therefore \sin \gamma = \frac{42}{240} \cdot \times \sin 54^\circ$$

$$= \cdot 14158.$$

$$\therefore \quad \gamma = 8^\circ\, 8',$$

and figure shows that r is upon the same side of the normal as OX. Also r is a tension, since OR is like OM.

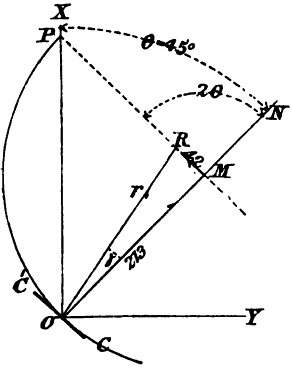


Fig. 39

63. In 62 find the intensity of the tangential stress on that plane through the point upon which the tangential stress is of greatest intensity.

The plane is that which is inclined at 45° to the axes of principal stress.

Since OMR is 90°, MR is the tangential component of OR,

$$r_t = \mathrm{MR} = \frac{p - q}{2}$$

$$= 42 \text{ lbs. per sq. in.}$$

64. In 62 find the obliquity to the plane of greatest principal stress of that plane, through the point, upon which the stress is more oblique than upon any other; also find the stress.

OM and MR being constant, the angle MOR has its greatest value when MRO is a right angle.

Construction—Upon OM describe a semi-circle; from M as centre, with radius MR, describe an arc cutting the semicircle in R; join OR.

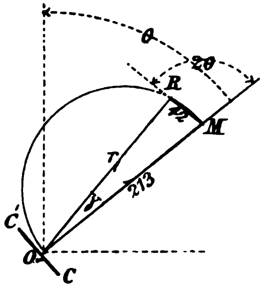


Fig. 40.

$$
\begin{aligned}
\cos 2\theta &= \cos \text{RMN} \\
&= -\cos \text{OMR} \\
&= -\frac{\text{MR}}{\text{OM}} = -\frac{p-q}{p+q} = -\frac{42}{213} \\
&= -\cdot 19718.
\end{aligned}
$$

$\therefore \quad 2\theta = 101^\circ\ 22'$ obtuse, cosine being negative.

$\therefore \quad \theta = 50^\circ\ 41'$, obliquity of CC′.

$$
\begin{aligned}
r^2 &= \text{OR}^2 = \text{OM}^2 - \text{MR}^2 \\
&= \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 \\
&= p \cdot q;
\end{aligned}
$$

$\therefore \quad r = \sqrt{p \cdot q} = \sqrt{43605}$

$= 208{\cdot}8$ lbs. per sq. in. of tension like OM.

and $\sin \gamma = \sin \text{RON}$

$$= \frac{\text{MR}}{\text{MO}} = \cdot 19718.$$

$\therefore \quad \gamma = 11^\circ\, 22'$, obliquity of r.

65. The principal stresses at a point being a tension of 300 lbs. and a thrust of 160 lbs. per square inch.

Find (*a*) The intensity, obliquity, and kind of stress on a plane through the point, inclined at 30° to the plane of greatest principal stress; (*b*) Find the intensity of tangential stress on the plane upon which that stress is greatest; and (*c*) Find the inclination to the plane of greatest principal stress of that plane upon which the stress is entirely tangential and the intensity thereof.

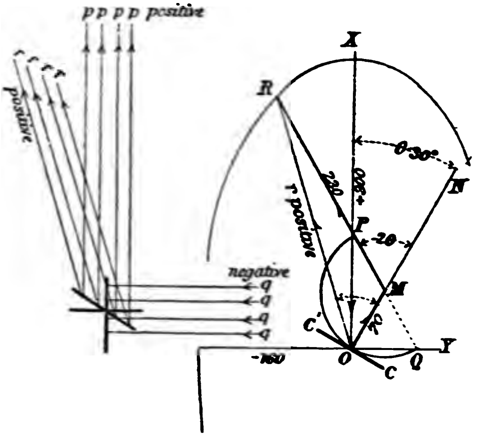


Fig. 41.

Data. $p = 300$; $q = -160$, considering a tension positive;

$$\therefore \frac{p+q}{2} = 70 \text{ tension like } p;$$

$$\text{and } \frac{p-q}{2} = 230 \text{ tension like } p.$$

(*a*) *Const.*—Draw ON at 30° to OX. Lay off OM = 70. From M as centre, with radius MO, describe semi-circle POQ. Lay off MR = 230. Then OR, the third side of the triangle *OMR*, taken in the opposite order, is the stress on CC′ in *direction and* intensity.

$$\begin{aligned}
\text{OR}^2 &= \text{OM}^2 + \text{MR}^2 - 2\text{OM} \,.\, \text{MR} \cos \text{OMR} \\
&= \text{OM}^2 + \text{MR}^2 + 2\text{OM} \,.\, \text{MR} \cos 2\theta. \\
r^2 &= 4900 + 52900 + 16100 \\
&= 73900, \\
r &= 272 \text{ lbs. per sq. in.}
\end{aligned}$$

and $$\frac{\sin \gamma}{\sin 2\theta} = \frac{\sin \text{ROM}}{\sin \text{RMO}} = \frac{\text{MR}}{\text{OR}}.$$

$$\therefore \quad \sin \gamma = \frac{230}{272} \sin 60^\circ = \cdot 7323.$$

$\therefore \quad \gamma = 47^\circ\ 5'$, being acute, OR is like OM, a tension.

(*b*) Take $\theta = 45^\circ$,

$$r_t = \text{MR} = 230 \text{ lbs.}$$

(*c*) On MR describe a semicircle, and from M with radius MO describe arc cutting it at O.

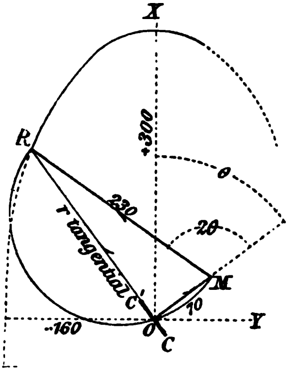


Fig. 42.

$$\begin{aligned}
\text{RMN} &= 2\theta. \\
\cos 2\theta &= \cos \text{RMN} \\
&= -\cos \text{RMO} \\
&= -\frac{\text{OM}}{\text{MR}} = -\frac{70}{230} \\
&= -\cdot 3044, \\
2\theta &= 107^\circ\, 44', \\
\theta &= 53^\circ\ 52', \text{ obliquity of plane, upon which the stress is entirely tangential.} \\
r^2 &= \text{OR}^2 \\
&= \text{MR}^2 - \text{OM}^2 \\
&= 52900 - 4900 \text{ or } \left(\frac{300+160}{2}\right)^2 - \left(\frac{300-160}{2}\right) \\
&= 48000 \qquad \text{or } 300 \times 160, \\
r &= 219 \qquad \text{or } \sqrt{p \,.\, q}.
\end{aligned}$$

Note that, though r is entirely tangential, it is less than r_t was in (*b*).

66. At same point as 65, find intensity, kind, and obliquity of a stress on a plane inclined at 85° to the plane of greatest principal stress.

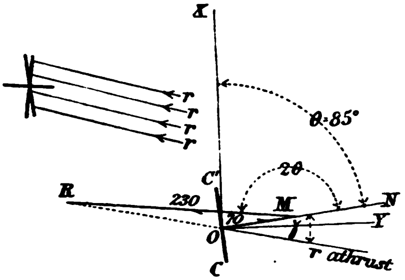


Fig. 43.

Since $\theta > 53° 52'$, $>$ the obliquity of plane upon which the stress was wholly tangential, OR will make with ON an angle greater than 90°, and OR will be unlike OM, and therefore a thrust.

Ans. $r = 161{\cdot}5$ lbs. per sq. in.,
$\gamma = 165° 41'$.

67. The principal stresses on AA′ and BB′ are thrusts of 60 lbs. per square inch. Find direction and intensity of the stress on a third plane CC′ inclined at 65° to AA′.

Ans. A thrust of 60 lbs. per square inch normal to CC′.

68. The principal stresses on AA′ and BB′ are of the equal intensity of ·34 lbs. per square inch, being a thrust on AA′ and a tension on BB′. Find the direction and intensity of the stress on a third plane CC′ inclined at 65° to AA′.

Ans. A tension of 34 lbs. per square inch, its direction being inclined at 65° upon the other side of OX from that to which ON is inclined.

69. The principal stresses on AA′ and BB′ at a point O are a thrust of 94 lbs. and a thrust of 26 lbs. Find kind, intensity, and obliquity of a stress on a third plane CC′ inclined at 65° to AA′. Using results of 67 and 68,

$r = 46{\cdot}2$ lbs. per sq. in. thrust,
$\gamma = 34° 19'$.

70. Two unlike principal stresses are: on AA′ a thrust of 146 and on BB′ a tension of 96 lbs. per square inch. Find the stress on CC′ a third plane inclined to AA′ at 50°.

$$p = 146 \text{ and } q = -96.$$

Half sum $\frac{p+q}{2}$ is a thrust like p.

Half diff. $\frac{p-q}{2}$ is a tension like q, since $\theta > 45°$.

(See p. 62.)

Ans. $r = 119{\cdot}22$ lbs. per sq. in. thrust,
$\gamma = 88° \, 5'$.

71. At a point within a solid the principal stresses are thrusts of 248 lbs. and 172 lbs. per square inch. Find the normal and tangential component stresses on a plane inclined at 15° to the plane of greatest principal stress,

$$r_t = (p - q) \sin \theta \cos \theta = 19 \text{ lbs. per sq. in.,}$$
$$r_n = p \, . \cos^2 \theta + q \sin^2 \theta = 243 \text{ lbs. per sq. in.}$$

These two results may be obtained with less labour from the formulæ

$$r_t = \frac{p-q}{2} \sin 2\,\theta,$$

$$r_n = \frac{p+q}{2} + \frac{p-q}{2} \cos 2\,\theta.$$

Given the intensities, obliquities, and kinds of the stresses upon any two planes at a point within a solid, find the principal stresses and their planes.

In the general problem we know of the triangle OMR (fig. 31, p. 65), the parts OR and γ for two separate positions of the plane CC′, and we also know that OM and MR are the *same* for both.

If the two given stresses be *alike* and *unequal*. Let r and r' be their intensities, and γ and γ' their obliquities upon their respective planes CC′ and DD′. Let r be greater than r'. Note that it is not necessary to have given the inclination to each other of CC′ and DD′.

Choose any line ON and draw OR $= r$, and making the angle NOR $= \gamma$, also draw OR′ $= r'$, and making the angle NOR′ $= \gamma'$. Join RR′, and from S the middle point of RR′ draw, at right angles to it, SM meeting ON at M. Then will MR $=$ MR′.

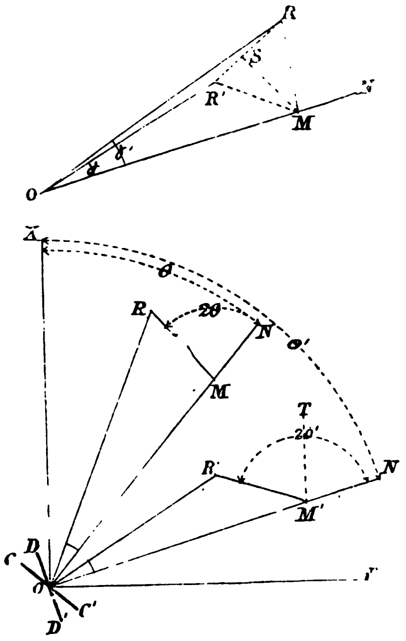


Fig. 44.

Thus we have found OM and MR to suit both data, and comparing the construction of the direct problem (p. 65), we have

$$\text{OM} = \frac{p+q}{2}$$

$$\text{and MR} = \frac{p-q}{2},$$

and therefore

$$p = \text{OM} + \text{MR}$$
$$\text{and } q = \text{OM} - \text{MR}.$$

Consider the triangle OM′R′ alone, and consider ON′ the normal to DD′: then R′M′N′ $= 2\,\theta'$, hence OX, drawn parallel to M′T (the bisector of R′M′N′) is the axis of greatest principal stress. Thus we have found the principal stresses p and q, and the position of their axis OX and OY relative to DD′ one of the given planes.

$$\text{Since} \qquad \text{R}'\text{MR} = \text{R}'\text{MN} - \text{RMN}$$
$$= 2\,\theta' - 2\,\theta,$$
$$\therefore \quad \text{RMS} = \theta' - \theta,$$

the inclination to each other of CC′ and DD′; hence if the other triangle OMR be moved round O through this angle, it and consequently CC′, to which ON is the normal, will also be in their proper positions with respect to the axes OX and OY.

This triangle might be further turned round O till ON is inclined at an angle XON $= \theta$ on the other side of OX, when CC′ would again be in a position for which the stress would be the same as given. This would increase the relative inclination of DD′ and CC′ by twice XON or by $2\,\theta$. Adding this to $\theta' - \theta$ gives $\theta' + \theta$. That is, the inclination of CC′ and DD′ to each other is

$$(\theta' - \theta) = \text{RMS on diagram,}$$
$$\text{or} \quad (\theta' + \theta) = \text{NMS on diagram,}$$

according as they lie on the same or on opposite sides of OX, the axis of principal stress.

If the two given stresses be *unlike* and *unequal*. Considering r the greater as positive, r' will be negative. Follow the same construction, only OR′ $= r'$ must be laid off from O in the *opposite* direction. Complete the figure as before, and we have from either figure—

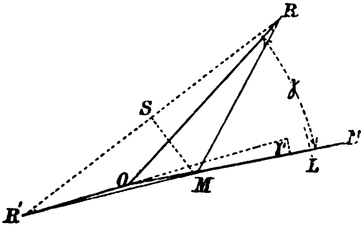


Fig. 45.

Trigonometrically.

$$\text{MR}^2 = \text{OM}^2 + \text{OR}^2 - 2\,\text{OM}\,.\,\text{OR}\cos\text{MOR},$$
$$\text{or MR}^2 = \text{OM}^2 + r^2 - 2\,\text{OM}\,.\,r\cos\gamma.$$

Similarly, $\text{MR}'^2 = \text{OM}^2 + r'^2 \mp 2\,\text{OM}\,.\,r'\cos\gamma'$ from figures 44 and 45 respectively.

Subtracting, $0 = r^2 - r'^2 - 2\,\text{OM}\,(r\cos\gamma \mp r'\cos\gamma')$,

$$\text{and } \therefore \frac{p+q}{2} = \text{OM} = \frac{r^2 - r'^2}{2\,(r\cos\gamma - r'\cos\gamma')} \quad \ldots\ldots\ldots\ldots\text{(A)}$$

r' to include its sign;

$$\left.\begin{aligned} \text{also } \frac{p-q}{2} = \text{MR} &= \sqrt{(\text{OM}^2 + r^2 - 2\,\text{OM}\,.\,r\cos\gamma)} \\ \text{or, } \qquad &= \sqrt{(\text{OM}^2 + r'^2 - 2\,\text{OM}\,.\,r'\cos\gamma')} \end{aligned}\right\}\ \text{(B)}$$

a known quantity when the value of OM is substituted from equation (A).

p and q are now obtained by adding and subtracting equations (A) and (B.

From R drop RL perpendicular to ON, then

$$\text{ML} = \text{OL} - \text{OM},$$

or
$$\text{MR} \cdot \cos \text{NMR} = \text{OR} \cdot \cos \text{ROM} - \text{OM},$$

or
$$\frac{p-q}{2} \cos 2\theta = r \cos \gamma - \frac{p+q}{2};$$

$$\therefore \qquad \cos 2\theta = \frac{2\,r \cos \gamma - p - q}{p - q} \quad \text{..........(C)}$$

This gives twice the obliquity of the axis of greatest principal stress to the given plane CC′ and similarly for DD′

$$^{*}\cos 2\,\theta' = \frac{2\,r' \cos \gamma' - p - q}{p - q}.$$

These three equations (A), (B) and (C), are the general solution of the inverse problem of the ellipse of stress. (A) and (B) give the intensities of the principal stresses, which will come out with signs showing whether they are like or unlike r, the greater of the given stresses.

In some particular cases the construction gives a much simpler figure from which the equations (A), (B) and (C) in their modified form are readily calculated.

Particular case (*a*). Given the intensities and common obliquity of a pair of *conjugate stresses* at a point; find the principal stresses, and position of the axes of principal stress. (Note, There are more than sufficient data.)

In this case $\gamma = \gamma'$ and R, S, and R′, are in one straight line with O.

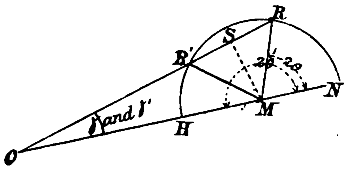


Fig. 46.

Draw any line ON, draw OR, making NOR $= \gamma = \gamma'$, and lay off OR $= r$ and OR′ $= r'$ in the same or opposite directions according as it is like or unlike r; and from S, the middle point of RR′, draw SM at right angles to it, meeting ON at M. Join M to R and R′.

---

* *Cos 2θ being* equal to cos RMO, may be calculated in terms of the sides *of the triangle* RMO when these have been already calculated.

Then
$$\frac{\mathrm{OS}}{\mathrm{OM}} = \cos \mathrm{MOS},$$

or
$$\mathrm{OM} = \frac{\mathrm{OS}}{\cos \mathrm{MOS}} = \frac{\frac{1}{2}(\mathrm{OR}+\mathrm{OR}')}{\cos \mathrm{MOS}},$$

or
$$\frac{p+q}{2} = \frac{r+r'}{2\cos\gamma}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Again,
$$\begin{aligned}
\mathrm{MR}^2 &= \mathrm{MS}^2 + \mathrm{RS}^2 \\
&= (\mathrm{OM}^2 - \mathrm{OS}^2) + \mathrm{RS}^2 \\
&= \mathrm{OM}^2 - (\mathrm{OS}^2 - \mathrm{RS}^2) \\
&= \mathrm{OM}^2 - \left\{\left(\frac{r+r'}{2}\right)^2 - \left(\frac{r-r'}{2}\right)^2\right\} \\
&= \mathrm{OM}^2 - rr',
\end{aligned}$$

(or substituting value of OM)

$$= \left(\frac{r+r'}{2\cos\gamma}\right)^2 - rr'.$$

$$\therefore\ \frac{p-q}{2} = \sqrt{\mathrm{OM}^2 - rr'} = \sqrt{\left\{\left(\frac{r+r'}{2\cos\gamma}\right)^2 - rr'\right\}}\quad\text{(B)}$$

and
$$\cos 2\theta = \frac{2r\cos\gamma - p - q}{p-q}\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}$$

as in general case,

$$\mathrm{NON}' = \theta' + \theta = \mathrm{NMS} = \mathrm{MSO} + \mathrm{MOS} = \frac{\pi}{2} + \gamma.$$

Hence, the angle between the two normals to the sections CC′ and DD′ (or the *obtuse* angle between CC′ and DD′) exceeds the obliquity by a right angle. This we know ought to be the case from the *definition* of conjugate stresses.

Practically, $\cos 2\theta = -\cos \mathrm{RMO}$ may be more easily calculated in terms of the sides of the triangle OMR when these have been already found.

From M as centre with radius MR describe the semicircle HR′RN; then

$$\begin{aligned} \text{OH} &= \text{OM} - \text{MR} = q, \\ \text{ON} &= \text{OM} + \text{MR} = p. \end{aligned}$$

But $\text{ON} \times \text{OH} = \text{OR} \times \text{OR}'$ (Euc. iii. 36),

or $$pq = rr' \qquad \text{(B}_1\text{)}$$

may be used instead of (B).

Particular case (*b*). Given the intensities and obliquities of the stresses on a pair of *rectangular planes*, find the principal stresses and the position of the axes of principal stress. (Note, There are more than sufficient data.)

If r and r' be like stresses,

Draw any line ON. Draw OR = R, making NOR = γ, also OR′ = r' making NOR′ = γ.

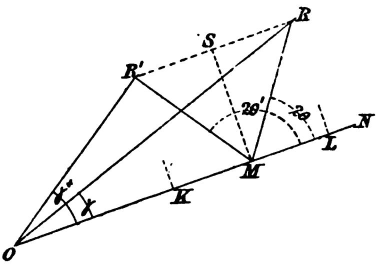


Fig. 47.

Complete the figure as before.

The given planes being at right angles are necessarily inclined upon opposite sides of the axis of principal stress; hence

$$\begin{aligned} \text{NMS} &= \text{inclination of given planes} \\ &= 90^\circ \end{aligned}$$

and RSR′ is parallel to ON.

$$\begin{aligned} \therefore \quad \text{MS} &= \text{RL} = r \sin \gamma, \\ \text{also,} \quad &= \text{R}'\text{K} = r' \sin \gamma', \\ \text{or} \quad r \sin \gamma &= r' \sin \gamma'. \end{aligned}$$

That is, the tangential components of r and r' are equal.

$$\text{OM} = \tfrac{1}{2}(\text{OL} + \text{OK}),$$

$$\therefore \quad \frac{p+q}{2} = \tfrac{1}{2}(r \cos \gamma + r' \cos \gamma'). \qquad \text{(A)}$$

That is, the sum of the principal stresses is equal to the sum of the normal components of r and r'. (Compare page 67.)

$$\begin{aligned} \text{MR}^2 &= \text{RS}^2 + \text{MS}^2 \\ &= \left(\frac{\text{OL} - \text{OK}}{2}\right)^2 + \text{MS}^2 \\ &= \frac{(r\cos\gamma - r'\cos\gamma')^2}{4} + r^2\sin^2\gamma. \end{aligned}$$

$$\therefore\ \frac{p-q}{2} = \sqrt{\left\{\frac{(r\cos\gamma - r'\cos\gamma')^2}{4} + r^2\sin^2\gamma\right\}} \text{......(B)}$$

$$\begin{aligned} \tan 2\theta &= \frac{\text{RL}}{\text{ML}} = \frac{\text{RL}}{\tfrac{1}{2}(\text{OL} - \text{OK})} \\ &= \frac{r\sin\gamma}{\tfrac{1}{2}(r\cos\gamma - r'\cos\gamma')} \\ &= \frac{2r\sin\gamma}{r\cos\gamma - r'\cos\gamma'} \text{..........................(C)} \end{aligned}$$

Putting $r_t = \text{MS} = r\sin\gamma = r'\sin\gamma' =$ the common value of the tangential components of r and r'; also

$r_n = \text{OL} = r\cos\gamma =$ norm. comp. of r,

$r_n' = \text{OK} = r'\cos\gamma' =$ norm. comp. of r',

the equations become

$$\frac{p+q}{2} = \frac{r_n + r'_n}{2} \text{..............................}(A_1)$$

$$\frac{p-q}{2} = \sqrt{\left\{\frac{(r_n - r'_n)^2}{4} + r_t^2\right\}} \text{.............}(B_1)$$

$$\text{and}\quad \tan 2\theta = \frac{2r_t}{r_n - r'_n} \text{............................}(C_1)$$

(Compare (C_1) with page 51.)

When r and r' are unlike stresses, consider r, the greater, as positive, then must OR′ be laid off in the opposite direction from O.

Now R′K = RL = r^t, the common tangential component of r, and r'; hence RR′ and KL *bisect* each other at S or M, which coincide.

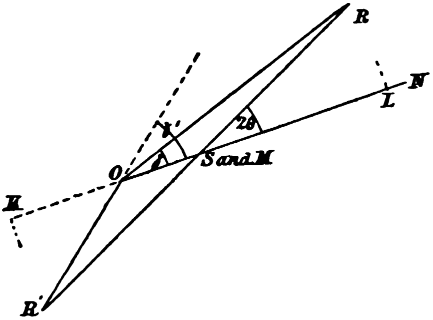


Fig. 48.

$$\therefore \quad \text{OM} = \tfrac{1}{2}(\text{OL} - \text{OK}),$$

$$\text{or} \quad \frac{p+q}{2} = \frac{r\cos\gamma - r'\cos\gamma'}{2} \quad \dots\dots\dots\dots (A_2)$$

$$\text{MR}^2 = \text{ML}^2 + \text{RL}^2$$

$$= \left(\frac{\text{OL} + \text{OK}}{2}\right)^2 + \text{RL}^2,$$

$$\text{or} \quad \frac{p-q}{2} = \sqrt{\left\{\frac{(r\cos\gamma + r'\cos\gamma')^2}{4} + r^2\sin^2\gamma\right\}} \dots\dots (B_2)$$

$$\tan 2\theta = \frac{\text{RL}}{\text{ML}} = \frac{\text{RL}}{\frac{1}{2}\text{LK}}$$

$$= \frac{\text{RL}}{\frac{1}{2}(\text{OL} + \text{OK})},$$

$$\text{or} \quad = \frac{2r\sin\gamma}{r\cos\gamma + r'\cos\gamma'} \dots\dots\dots\dots (C_2)$$

These three equations (A_2), (B_2), and (C_2) are identical *with (A), (B)* and (C) with ($-r'$) substituted for r'.

Examples.

72. If from external conditions it be known that the stresses on two planes at a point in a solid are thrusts of 54 and 30 lbs. per square inch, and inclined at 10° and 26° respectively to the normals to these planes,—find the principal stresses at that point; the position of the axis of greater principal stress relative to the first plane; and the inclination of the two planes to each other.

Make $\text{NOR} = \gamma = 10^\circ$,
and $\text{NOR}' = \gamma' = 26^\circ$.
Lay off $\text{OR} = r = 54$,
and $\text{OR}' = r' = 30$.

Join RR′, bisect it in S, draw SM at right angles to RR′, meeting ON at M, complete figure.

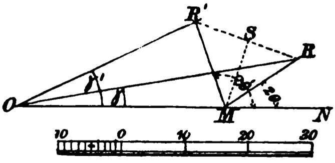


Fig. 49.

Then $\frac{p+q}{2} = \text{OM}$ and $\frac{p-q}{2} = \text{MR} = \text{MR}'$,

or $p = (\text{OM} + \text{MR})$ and $q = (\text{OM} - \text{MR})$, also $2\theta = \text{NMR}$.

Trigonometrically.

$$\text{MR}^2 = \text{OM}^2 + \text{OR}^2 - 2\text{OM}\,.\,\text{OR}\cos\text{MOR},$$

or $$\text{MR}^2 = \text{OM}^2 + r^2 - 2\text{OM}\,.\,r\cos\gamma.$$

Similarly, $$\text{MR}'^2 = \text{OM}^2 + r'^2 - 2\text{OM}\,.\,r'\cos\gamma'.$$

∴ Subtracting, $$0 = r^2 - r'^2 - 2\text{OM}(r\cos\gamma - r'\cos\gamma').$$

$$\therefore\ \text{OM} = \frac{r^2 - r'^2}{2(r\cos\gamma - r'\cos\gamma')},$$

or $$\frac{p+q}{2} = \frac{2016}{52{\cdot}43} = 38{\cdot}45 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

$$\text{MR}^2 = (38{\cdot}45)^2 + (54)^2 - 2 \times 38{\cdot}45 \times 54\cos 10^\circ$$
$$= 1478{\cdot}4 + 2916 - 4088{\cdot}8 = 305{\cdot}6.$$

$$\therefore\ \frac{p-q}{2} = \sqrt{305{\cdot}6} = 17{\cdot}48 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$$

The principal stresses are—

$$p=\frac{p+q}{2}+\frac{p-q}{2}=55{\cdot}93 \text{ lbs. per sq. in. thrust, like } r.$$

$$q=\frac{p+q}{2}-\frac{p-q}{2}=20{\cdot}97 \text{ lbs. per sq. in., thrust being } +.$$

Drop RL perpendicular to ON.

$$\text{ML}=\text{OL}-\text{OM},$$

or $\text{MR} \cos \text{LMR}=\text{OR} \cos \text{LOR}-\text{OM}.$

$$\therefore \quad \frac{p-q}{2}\cos 2\theta=r\cos\gamma-\frac{p+q}{2} \qquad \text{(C)}$$

$$\cos 2\theta=\frac{53{\cdot}179-38{\cdot}45}{17{\cdot}48}={\cdot}8426.$$

$$\therefore \quad 2\theta=32^\circ\ 35'.$$

$$\therefore \quad \theta=16^\circ\ 17\tfrac{1}{2}'=\text{XON},$$

the inclination of OX, the axis of greatest principal stress, to ON, the normal to the plane for which r was given.

Similarly, $\cos 2\theta'=-{\cdot}6573,$

$$\therefore \quad 2\theta'=131^\circ\ 6' \text{ (obtuse for}-\text{)},$$

$$\therefore \quad \theta'=65^\circ\ 33', \text{ inclination XON}'.$$

And inclination of the two planes to each other—

$$\text{NON}'=\text{RMS}=(\theta'-\theta)=49^\circ\ 15\tfrac{1}{2}',$$

$$\text{or} \quad =\text{NMS}=(\theta'+\theta)=81^\circ\ 50\tfrac{1}{2}',$$

according as they are on the same or opposite sides of OX.

73. Knowing that at a point within a solid there is, on some one plane, a thrust of 84 lbs. per square inch of obliquity 5°, and on another a tension of 24 lbs. per square inch of obliquity 20°, find the principal stresses and the angle *made by the* axis of greatest principal stress with the *normal to first* plane.

Consider a thrust positive (see 2nd fig. inverse problem).

$$r=84 \text{ and } \gamma=5^\circ;\ r'=-24 \text{ and } \gamma'=20^\circ.$$

$\frac{p+q}{2}=\text{OM}$ is a thrust, like r

$$=\frac{r^2-r'^2}{2(r\cos\gamma-r'\cos\gamma')}=30{\cdot}5\ldots\ldots\ldots\ldots\ldots(\text{A})$$

$\frac{p-q}{2}=\text{MR}$ is a thrust like r.

$$\text{MR}^2=\text{OM}^2+r^2-2\text{OM}\,.\,r\cos\gamma$$
$$=930{\cdot}25+7056-5104{\cdot}48.$$

$$\frac{p-q}{2}=\text{MR}=\sqrt{2881{\cdot}77}=53{\cdot}7\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{B})$$

∴ by adding and subtracting (A) and (B),

$$p=84{\cdot}2 \text{ and } q=-23{\cdot}2,$$

or the principal stresses are a thrust of 84·2 and a tension of 23·2 lbs. per square inch respectively.

$$\cos 2\theta=\frac{2r\cos\gamma-p-q}{p-q}$$
$$=\frac{167{\cdot}36-61}{107{\cdot}4}=\cdot9903.$$

$\therefore\ 2\theta=7^\circ\ 59'$ and $\theta=3^\circ\ 59\frac{1}{2}'$, inclination to axis OX.

74. The stresses on two planes at a point within a solid are 240 at an obliquity of 8°, and 254·5 at 1° 5′. Find the principal stresses and the obliquities of these planes to the axes of principal stress. (Note, these are the planes BB′, Ex. 57, and GG′, Ex. 59; also principal stresses are calculated Ex. 58.)

$$\left.\begin{array}{l}\frac{p+q}{2}=\text{OM}=213{\cdot}5\\ \frac{p-q}{2}=\text{MR}=\ 41{\cdot}26\end{array}\right\}\begin{array}{l}\therefore p=254{\cdot}76,\\ \therefore q=172{\cdot}24,\end{array}$$

$\cos 2\theta=\cdot9926$ or $2\theta=7^\circ\ 1'$. $\therefore \theta=3^\circ\ 30\frac{1}{2}$.

Or geometrically—describe semicircle HR′RN.

$$\text{ON}=\text{OM}+\text{MR}=p \text{ and } \text{OH}=\text{OM}-\text{MR}=q.$$

$$\text{ON}\,.\,\text{OH}=\text{OR}\,.\,\text{OR}' \text{ (Euc. III. 36)}$$

$$\text{or } p\,.\,q=rr' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B}'\text{)}$$

$$=1200.$$

$$\text{Now} \quad p+q=71{\cdot}1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

$$\therefore \quad p^2+2pq+q^2=5055{\cdot}2,$$

$$\text{but (B),} \quad 4pq \quad =4800.$$

$$\therefore \quad p^2-2pq+q^2=255{\cdot}2.$$

$$\therefore \quad p-q \quad =16,$$

adding to and subtracting from (A),

$$\therefore \quad 2p=71{\cdot}1+16 \text{ and } 2q=71{\cdot}1-16.$$

$$\therefore \quad p=43{\cdot}55 \quad \text{and} \quad q=27{\cdot}55.$$

75. At a point within a solid, on one plane, there is a tension of 272 lbs. per square inch, of obliquity 47° 5′, and on another a thrust of 161·5 lbs. per square inch, of obliquity 15° 25′. Find the principal stresses and the angles which the normals to these planes make with the axis of greatest principal stress.

$$\left.\begin{array}{l} \dfrac{p+q}{2}=\text{OM}=70{\cdot}2 \\ \dfrac{p-q}{2}=\text{MR}=230 \end{array}\right\} \begin{array}{l} \therefore p = 300{\cdot}2 \text{ tension,} \\ \therefore q = -\ 159{\cdot}8 \text{ thrust,} \end{array}$$

$$\cos 2\theta = {\cdot}5, \text{ or } 2\theta = 60^\circ \qquad \therefore \theta = 30^\circ,$$

$$\cos 2\theta' = -{\cdot}982 \text{ or } 2\theta' = 169^\circ\ 8'. \quad \therefore \theta' = 84^\circ\ 34'$$

$$= 85^\circ \text{ nearly.}$$

∴ *Inc. between* planes, $(\theta' - \theta) = 55^\circ$. Compare Examples *65 and 66.*

76. At a point within a solid a pair of conjugate stresses are thrusts of 40 and 30 lbs. per square inch, and their common obliquity is 10°. Find the principal stresses and the angle which normal to plane of greater conjugate stress makes with the axis of greatest principal stress.

Draw OR, making NOR $= \gamma = \gamma' = 10°$; lay off OR $= r = 40$ and OR′ $= r' = 30$. Bisect RR′ in S, draw SM perpendicular to RR′; complete figure. Then $\frac{p+q}{2} =$ OM, and $\frac{p-q}{2} =$ MR, and $2\theta =$ RMN.

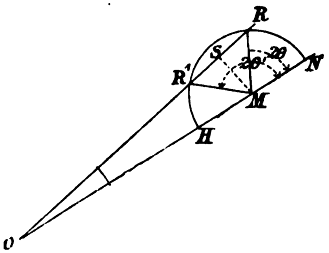


Fig. 50.

$$\frac{\text{OS}}{\text{OM}} = \cos\gamma,$$

$$\therefore \quad \text{OM} = \frac{\text{OS}}{\cos\gamma} = \frac{\frac{1}{2}(r+r')}{\cos\gamma},$$

$$\text{or} \quad \frac{p+q}{2} = \frac{35}{\cdot 9848} = 35{\cdot}55 \text{ ....}(\text{A})$$

$$\begin{aligned}\text{MR}^2 &= \text{MS}^2 + \text{RS}^2\\ &= (\text{OM}^2 - \text{OS}^2) + \text{RS}^2\\ &= \text{OM}^2 - (\text{OS}^2 - \text{RS}^2)\\ &= \text{OM}^2 - \left\{\left(\frac{r+r'}{2}\right)^2 - \left(\frac{r-r'}{2}\right)^2\right\}\\ &= \text{OM}^2 - rr'.\end{aligned}$$

$$\therefore \quad \frac{p-q}{2} = \sqrt{(1263{\cdot}8 - 1200)} = 8 \text{ ....}(\text{B})$$

Adding and subtracting (A) and (B),

$$p = 43{\cdot}5 \text{ a thrust, and } q = 27{\cdot}5 \text{ a thrust,}$$

$$\cos 2\theta = \frac{78{\cdot}8 - 71}{16} = \cdot 49 \text{ ....}(\text{C})$$

$$\therefore \quad 2\theta = 60°\ 40' \text{ and } \theta = 30°\ 20'.$$

77. At a point within a solid, a pair of conjugate stresses are 182 (tension) and 116 (thrust), common obliquity 30°. Find the principal stresses and the position of axes.

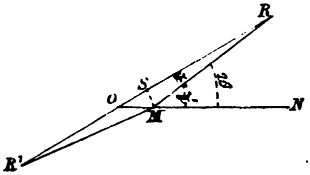


Fig. 51.

r' is negative

$$\frac{p+q}{2} = \text{OM} = 38{\cdot}14,$$

$$\frac{p-q}{2} = \text{MR} = 150{\cdot}3,$$

$\therefore p = 188{\cdot}4$ (thrust) and $q = -112{\cdot}2$ (tension),

$\cos 2\theta = {\cdot}7947 \qquad \therefore \theta = 18° 41'.$

78. The stresses on two planes at right angles to each other being thrusts of 240 and 193 lbs. per square inch of obliquities respectively 8° and 10°. Find the principal stresses also and their axes.

$r_n = r \cos\gamma$ and $r'_n = r' \cos\gamma'$; also $r_t = r \sin\gamma = r' \sin\gamma'$
$= 237{\cdot}6 \qquad\qquad = 190 \qquad\qquad = 33{\cdot}4$

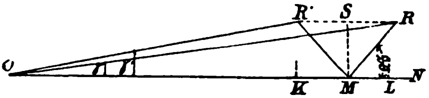


Fig. 52.

$$\frac{p+q}{2} = \text{OM} = \frac{r_n + r'_n}{2}$$
$$= 213{\cdot}8,$$
$$\frac{p-q}{2} = \text{MR} = \sqrt{\left\{\left(\frac{r_n - r'_n}{2}\right)^2 + r_t^2\right\}}$$
$$= \sqrt{(566 + 1115)}$$
$$= 41,$$

$\therefore p = 254{\cdot}8$ and $q = 172{\cdot}8,$

$$\tan 2\theta = \frac{2r_t}{r_n - r'_n} = 1{\cdot}4034,$$

$\therefore \; 2\theta = 54° 32' \quad \therefore \theta = 27° 16'.$

(See Ex. 74 and 57.)

79. In last, had the 193 been a tension. Find the principal stresses.

$$r = 240, \text{ and } r' = -193,$$
$$r_n = 237{\cdot}6, \quad r'_n = -190,$$

also $(r_t)^2$ is always positive,

$$\frac{p+q}{2} = \text{OM} = \frac{r_n + r'_n}{2} = 23{\cdot}8,$$
$$\frac{p-q}{2} = \text{MR} = \sqrt{\left\{\left(\frac{r_n - r'_n}{2}\right)^2 + r_t^2\right\}}$$
$$= \sqrt{\{(213{\cdot}8)^2 + (33{\cdot}4)^2\}} = 216{\cdot}3,$$
$$\therefore \quad p = 240{\cdot}1, \text{ a thrust;}$$
$$\text{and} \quad q = -192{\cdot}5, \text{ a tension.}$$
$$\tan 2\theta = \frac{2r_t}{r_n - r'_n} = \frac{66{\cdot}8}{427{\cdot}6} = {\cdot}1562,$$
$$\therefore \quad 2\theta = 8^\circ\,53' \qquad \therefore\ \theta = 4^\circ\,26'.$$

Application of Ellipse of Stress to Stability of Earthwork.

Loose earth, built up into a mass on a horizontal plane, will only remain in equilibrium with its faces at slopes, whose inclinations to the horizontal plane are less than an angle ϕ. If the earth be heaped up till the slope is greater, it will *run* till the slope is at greatest ϕ. Moist and compressed masses of earth can be massed up into a heap with slopes greater than ϕ, and will remain in equilibrium for some time, but will ultimately crumble down till the slopes do not exceed ϕ, The surface soil, which is in a compressed state, may be cut away, leaving banks with slopes much greater than ϕ. These banks will only remain in equilibrium for a time. Slips will occur till ultimately the slopes are not greater than ϕ.

This angle ϕ, which is the greatest inclination (of the slopes to the *horizontal* plane) at which a mass of earth will remain in equilibrium, is called the *angle of repose.* It has different values for different kinds of earth, and also different values for the same earth kept at different degrees of moistness. Average values of ϕ for different kinds of earth have been ascertained by experiment and observation, and are tabulated.

If two particles of earth are pressed together by a pair of equal thrusts p and p' normal to their surface of contact, it requires a pair of equal thrusts q and q' tangential to that surface to make them slide upon each other. For the same material, when q is just sufficient to make them slide, it is a constant fraction of p. The fraction which q requires to be of p just to cause slipping is called the *co-efficient of friction* for that material. Hence the co-efficient of friction

$$\mu = \frac{q}{p}.$$

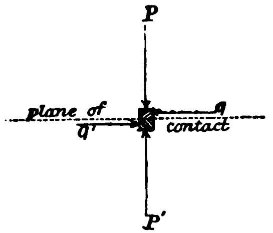


Fig. 53.

The figure is section of two troughs enclosing earth, and pressed together with a thrust of intensity p normal to MN, the plane where the troughs are just not in contact, and P is the amount of this thrust. A thrust of intensity q tangential to the plane MN tends to cause the earth to slide in two parts along MN, also Q is the amount of this thrust. If Q be just sufficient to cause slipping along MN, then the co-efficient of friction of the earth is

$$\mu = \frac{Q}{P}.$$

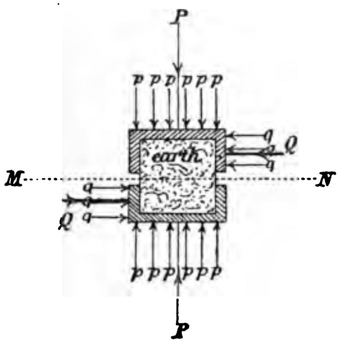


Fig. 54.

If on AB and CD there be a thrust of intensity p inclined at an angle ϕ to the normal, we know that for equilibrium of the prism ABCD there must be a stress q upon the faces AC and BD, whose tangential component equals that of p, but as far as stability along the plane MN is concerned we may neglect q, whose normal components destroy each other through the material of the trough, and the tangential ones are at right angles to MN. Considering the components of P, the amount of p, we have P $\cos\phi$ normal to MN. If slipping is just about to take place, then

$$\mu = \frac{p \sin \phi}{p \cos \phi} = \tan \phi.$$

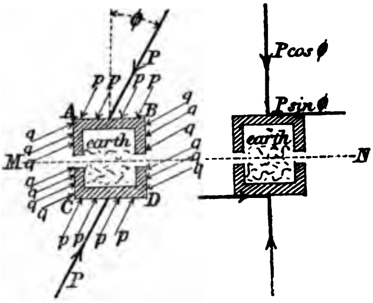


Fig. 55.

It is apparent that ϕ is the same angle we were before considering, for, if P be due to the weight of the material, the figure ought to be turned till the direction of P is vertical, when MN the plane of slipping will be inclined at ϕ to the horizontal. The relation between the co-efficient of friction and the angle of repose is

$$\mu = \tan \phi.$$

NOTE.—If it were not upon the supposition that the two troughs (being very rigid compared to the earth) transmitted the equal and opposite forces tangential to MN without causing lateral compression of the earth, we could not neglect q. From this result we learn that the tendency to slip along the plane MN, due to p, depends entirely upon the obliquity of p, and not at all upon its intensity. Thus, if p be inclined at an angle less than ϕ, slipping will not occur though p be ever so great: but, if p be inclined at an angle greater than ϕ, slipping will take place, though p be ever so small.

Consider now the equilibrium of a small prism at a point within a mass of earth in a compound state of strain. The earth will have a tendency to slip along any plane through the point (as there is no artificial envelope), except along the planes of principal stress at the point; and the tendency to slip will be greater along the plane upon which the resultant stress is more oblique, and greatest along the pair of planes upon which the *resultant stress is most oblique*, it being of no consequence how intense the stresses upon these various planes may be, but only how oblique. If the stresses upon the pair of planes, for which the resultant stress is more oblique than that upon any other plane through the point, be themselves less oblique than ϕ, no slipping will occur upon any plane through that point; but if more oblique than ϕ, slipping will take place along one or both of those planes.

The condition of equilibrium of a mass of earth in a compound state of strain is that at every point the obliquity of the stress on the plane upon which, of all others through the point, the resultant stress is most oblique, shall itself not be greater than ϕ.

Since earth can only sustain thrusts, the principal stresses at a point will be both thrusts which excludes case (*c*), and if γ be the obliquity of the resultant stress upon the plane through the point upon which the stress is most oblique, then by case (*d*) (page 73),

$$\sin\gamma = \frac{p-q}{p+q}.$$

$$\therefore \quad \frac{p}{q} = \frac{1+\sin\gamma}{1-\sin\gamma}.$$

By increasing γ the numerator of the term on right-hand side of equation increases, while the denominator decreases, and so the ratio $\frac{p}{q}$ increases. But ϕ is the greatest value of γ for which equilibrium is just possible.

$$\therefore \quad \frac{p}{q} = \frac{1+\sin\phi}{1-\sin\phi}$$

is the greatest ratio of p to q consistent with equilibrium; *hence*—

The condition of equilibrium of a mass of earth is most conveniently stated thus: that at every point the ratio of the greater to the lesser principal stress shall not exceed that of $(1 + \sin\phi)$ *to* $(1 - \sin\phi)$.

Or geometrically.

Let $\text{OM} = \dfrac{p+q}{2}$.

Make $\text{MOR} = \phi$.

Drop MR perpendicular to OR.

Describe the semicircle HRN.

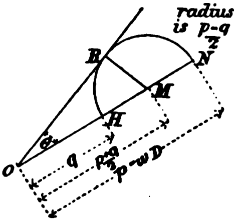


Fig. 56.

Because MOR = obliquity of thrust on plane which sustains most oblique strain,

and ORM = 90°.

$$\text{MR} = \frac{p-q}{2}. \quad \text{See case } (d) \text{ (page 73).}$$

$$\therefore \quad \text{ON} = (\text{OM} + \text{MR}) = p,$$

$$\text{and} \quad \text{OH} = (\text{OM} - \text{MR}) = q.$$

$$\therefore \quad \frac{p}{q} = \frac{\text{ON}}{\text{OH}} = \frac{\text{OM} + \text{MR}}{\text{OM} - \text{MR}}$$

$$= \frac{\text{OM} + \text{OM}\sin\phi}{\text{OM} - \text{OM}\sin\phi}$$

$$= \frac{1 + \sin\phi}{1 - \sin\phi}.$$

For earth whose upper surface is horizontal, the vertical stress due to the weight and the horizontal stress are for *all* points the principal stresses, and their intensities are the same for all points on the same horizontal plane. Generally the vertical is the greater principal stress in any ratio not exceeding the above, whenever it exceeds the horizontal thrust by a greater ratio the earth *spreads*. But the horizontal thrust may be artificially increased till it exceeds the vertical in any ratio not exceeding the above. Whenever it exceeds the vertical by a greater ratio, the earth *heaves up*.

The *third axis* of principal stress, which we are all along neglecting, is also horizontal. When the earth is in horizontal layers with a horizontal surface, all vertical planes are symmetrical, and the three planes of principal stress are *any two* vertical planes at right angles to each other and the horizontal plane. The stress on the two vertical planes being equal, the *ellipsoid of stress becomes a spheroid.* When, however, the horizontal thrust on one vertical plane is artificially increased, that plane becomes *one* of the planes of principal stress, and the stress may be different on all three.

Earth in horizontal layers loaded with its own weight to find the pressure against a retaining wall with vertical face.

Let

w = weight in lbs. of a cub. ft. of earth,

ϕ = its angle of repose,

D = depth of cutting.

Consider a layer 1 foot thick normal to paper, and choose a small rectangular prism at depth x feet.

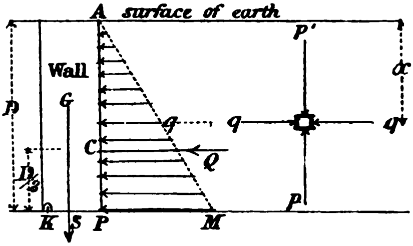


Fig. 57.

Let p = intensity of vertical pressure at depth x, in lbs. per square foot.

= weight of a volume of earth one square foot, in section and x ft. high,

= wx lbs.

If q = least horizontal stress which will give equilibrium, we have

$$\frac{p}{q} = \frac{1 + \sin\phi}{1 - \sin\phi},$$

$$\therefore \quad q = \frac{1 - \sin\phi}{1 + \sin\phi} \cdot p$$

$$= \frac{1 - \sin\phi}{1 + \sin\phi} w \, . \, x \text{ lbs. per sq. ft.}$$

= intensity of pressure on wall at the depth x.

On the right side of equation all is constant but x, hence

q is proportional to x, is zero at the top,

and uniformly increases to

$$q = \frac{1 - \sin \phi}{1 + \sin \phi} w \,.\, \text{D at the bottom,}$$

and therefore

$$\frac{1 - \sin \phi}{1 + \sin \phi} \cdot \frac{w\text{D}}{2} = \text{average intensity of pressure upon wall.}$$

And the area exposed to this pressure is D square feet. Hence the total pressure on wall is

$$\begin{aligned} \text{Q} &= \text{average intensity of pressure} \times \text{area} \\ &= \frac{1 - \sin \phi}{1 + \sin \phi} \cdot \frac{w\text{D}^2}{2} \text{ lbs.} \end{aligned}$$

This tends to make the wall slide as a whole along MB; for equilibrium the weight of the wall, multiplied by the coefficient of friction at the bed joint there, must be greater than Q.

If BM be laid off to represent the horizontal pressure at B, and M be joined to A, then MA gives the horizontal thrusts at all points as shown by arrows; Q, the resultant of all these, is horizontal and passes through the centre of gravity of the triangle ABM, it therefore acts at a point C called the centre of pressure, and

$$\text{BC} = \tfrac{1}{3}\,\text{BA} = \frac{\text{D}}{3}.$$

Q tends to overturn the wall with a moment,

$$\begin{aligned} \text{M} &= \text{Q} \times \text{leverage about B,} \\ &= \text{Q} \cdot \frac{\text{D}}{3}, \\ &= \frac{1 - \sin \phi}{1 + \sin \phi} \cdot \frac{w\text{D}^3}{6} \text{ foot-lbs.} \end{aligned}$$

Let K be the centre of the vertical pressure due to the weight of the wall and horizontal pressure of earth at the bed *joint at* M; *also* let the vertical line drawn through G, the centre of gravity of the wall, cut the joint at S, then for equilibrium the moment,

the weight of wall × leverage KS,

must be greater than the overturning moment, M.

It is generally sufficient to ascertain if this lowest bed joint be stable: but for some forms of wall it is necessary to go through all calculations for each bed joint considered in turn as bottom of wall.

In a wall of uniform thickness throughout its height the weight increases as D, whereas the force Q increases as D^2, and the lowest bed joint is most severely taxed. Similarly, for overturning, KS being constant, the product, KS × weight of wall, increases as D while M increases as D^3. K would be the extreme outside of the wall if the material were perfectly strong. For stone retaining walls SK is $\frac{3}{8}$ths of the half thickness.

Examples.

80. The weight of a certain earth is 120 lbs. per cubic foot, its angle of repose 25°. It is spread in horizontal layers. Find the average intensity of the pressure against a retaining wall with vertical face and 4 feet in depth. Also, find total pressure against a slice of wall 1 foot in the direction of the length of the wall and the overturning moment of the earth about the lowest point.

$$p = 4\,w = 480 \text{ lbs per square foot,}$$

$$q = \frac{1 - \sin\phi}{1 + \sin\phi} \cdot p = 194{\cdot}8 \text{ per square foot,}$$

$$\text{Average pressure} = \tfrac{1}{2}\,q = 97{\cdot}4 \text{ per square foot,}$$

$$\text{Total pressure } Q = 97{\cdot}4 \text{ lbs. per square foot} \times 4 \text{ square feet,}$$

$$= 389{\cdot}6 \text{ lbs.}$$

$$\text{Overturning moment, } M = Q \text{ lbs.} \times \tfrac{4}{3} \text{ ft.} = 519{\cdot}5 \text{ ft.-lbs.}$$

81. Gravel is heaped against a vertical wall to a height of 3 feet; weight of gravel 94 lbs. per cubic foot; angle of repose, 38°. Find horizontal thrust per lineal foot of wall, also overturning moment.

$$Q = 100{\cdot}5 \text{ lbs.}; \; M = 100{\cdot}5 \text{ ft.-lbs.}$$

82. A ditch 6 feet deep is cut with vertical faces in clay. These are shored up with boards, a strut being put across from board to board 2 feet from bottom at intervals of 5 feet apart. The co-efficient of friction of the moist clay is ·287, and it weighs 120 lbs. per cubic foot. Find the thrust on a strut, also find the greatest thrust which might be put upon the struts before the adjoining earth would heave up.

Since $\tan\phi = \cdot 287$, $\therefore \sin\phi = \cdot 276$,

$\therefore \quad Q = 1225{\cdot}5$ lbs. per lineal foot.

Thrust per strut $= 6127{\cdot}5$ lbs., just to prevent earth from falling in.

Greatest thrust which might be artificially put upon each strut before earth would heave up $= 19{,}029$ lbs.

Depth to which the foundation of a wall must, at least, be sunk in earth laid in horizontal layers consistent with equilibrium of earth.

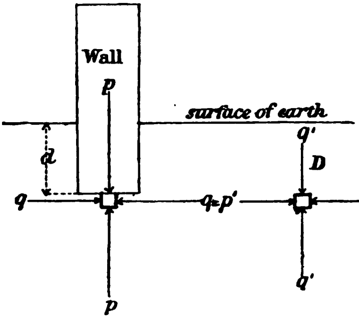


Fig. 58.

Consider one lineal foot of wall, normal to paper.

$V =$ vol. of wall in cub. ft.,

$W =$ wt. of wall per „

$h =$ height of wall in feet,

$b =$ breadth of wall „

$d =$ required depth of found,

$w =$ wt. per cubic feet of earth,

$\phi =$ its angle of repose.

When the wall has just stopped subsiding, the earth on each side is on the point of heaving up, so at the horizontal layer at the depth of d, for points in contact with the bottom of found—p exceeds q in the greatest possible limit, that earth being on the point of spreading,

$$\text{or } \frac{p}{q} = \frac{1 + \sin\phi}{1 - \sin\phi},$$

while, for points just clear of it, p' exceeds q' in that limit.

$$\therefore \qquad \frac{p'}{q'} = \frac{1 + \sin \phi}{1 - \sin \phi},$$

by multiplication $$\frac{pp'}{qq'} = \left(\frac{1 + \sin \phi}{1 - \sin \phi}\right)^2.$$

Now $p' = q$, being horizontal thrust on same horizontal layer, cancel these and substitute the values

$$p = \frac{\text{weight of wall}}{\text{area exposed to } p} = \frac{WV}{b},$$

$$q' = \text{weight of column of earth} = wd,$$

hence we have $$\frac{WV}{bwd} = \left(\frac{1 + \sin \phi}{1 - \sin \phi}\right)^2,$$

$$\therefore \quad d = \frac{WV}{wb}\left(\frac{1 - \sin \phi}{1 + \sin \phi}\right)^2 \text{ feet.}$$

83. A wall 10 ft. high and 2 ft. thick, and weighing 144 lbs. per cub. ft., is founded in earth 112 lbs. per cub. ft., and whose angle of repose is 32°. Find least depth of found.

$p =$ int. of vert. pressure below bottom of found

$= 144 \times 10 = 1440$ lbs. per sq. ft.,

$q' =$ Int. of vert. pressure at same depth clear of found

$= 112 \,.\, d,$

but $$\frac{q'}{p} = \left(\frac{1 - \sin \phi}{1 + \sin \phi}\right)^2,$$

$$\therefore \quad \frac{d \,.\, 112}{1440} = \cdot 094. \quad \therefore d = 1{\cdot}21 \text{ ft.}$$

NOTE.—The height of wall above ground is $10 - d = 8{\cdot}79$ ft.

84. An iron column is to bear a weight of 20 tons; the found is a stone 3 ft. square on bed, sunk in earth weighing

120 lbs. per cub. ft., angle of repose 27°. Find least depth to which it must be sunk for equilibrium.

$$p = \frac{44800 \text{ lbs.}}{9 \text{ sq. ft.}} = 4978 \text{ lbs. per sq. ft.}$$

$$q' = 120\,d,$$

$$\text{but } \frac{q'}{p} = \left(\frac{1 - \sin \phi}{1 + \sin \phi}\right)^2. \quad \therefore\ d = \frac{4978}{120} \times \cdot 141 = 6 \text{ ft.}$$

85. A brick wall, allowing for openings, weighs 42,000 lbs. per rood of 36 sq. feet (on an average one brick and a half), and stands 45 feet above ground; the foundation is to widen to four bricks at bottom. Find depth of found in clay weighing 130 lbs. per cub. ft. (angle of repose 27°). 1st, Neglect weight of unknown found.

$$WV = \text{wt. of 1 lineal foot of wall} = 4667 \text{ lbs.}$$

$$p = \frac{WV}{b} = \frac{\text{wt. of lin. ft.}}{\text{area of base}} = \frac{4667 \text{ lbs.}}{3 \text{ sq. ft.}} = 1556 \text{ lbs. per sq. ft.}$$

$$q' = 130\,.\,d,$$

$$\text{but } \frac{q'}{p} = \left(\frac{1 - \sin 27^\circ}{1 + \sin 27^\circ}\right)^2,$$

$$\text{or } \frac{130\,.\,d}{1556} = \cdot 141. \quad \therefore\ d = 1\cdot 7 \text{ ft. least depth.}$$

Say 2 ft. deep by an average of 3 bricks thick, *i.e.*, $4\frac{1}{2}$ cub. ft. per lineal ft., at 125 lbs., gives extra weight of 563 lbs. Adding this,

$$\therefore\ \text{wv} = 5230 \text{ lbs.} \quad \therefore\ p = 1743,$$

$$\text{and } d = \frac{1743}{130} \times \cdot 141 = 2 \text{ ft.}$$

For safety this would require to be increased.

Earth spread in layers at a uniform slope, and loaded with its own weight, to find the pressure against a retaining wall with vertical face.

The simplest (commonest in practice) case is when the vertical face of wall is at right angles to the section showing greatest declivity of free surface. Let the paper be that section; then AB is the trace of the upper surface, and γ is its *greatest* inclination to the horizon.

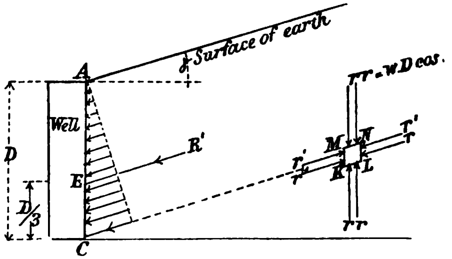


Fig. 59.

This inclination must be less than the angle of repose, or the earth would run over the wall. In an *extreme* case they may be equal.

Generally, $\gamma < \phi$.

Taking a slice one foot normal to paper.

Suppose the earth to be spread behind the wall in layers sloping at the angle γ, consider a small parallelopiped in the layer of depth D having vertical faces. At this depth, D, the intensity in lbs. per sq. ft. of the vertical pressure due to the weight of earth above, on *a horizontal surface,* would be the weight of a cub. ft. of earth multiplied by the depth D in feet. Hence

$$w \, . \, \mathrm{D} \text{ lbs. per sq. ft.}$$

= intensity of vertical pressure on parallelopiped had its *surface* been horizontal.

But the sloping surface MN is greater than the corresponding horizontal surface, that supports the same earth; so the vertical stress thereon will be less than $w\text{D}$, page 40, and will be

$$r = w\text{D} \cos \gamma \text{ lbs. per sq. ft.}$$

This is the intensity of the pressure upon the faces MN and KL, and its direction is vertical and therefore parallel to any pair of vertical faces of the parallelopiped; hence the pressure on any pair of vertical faces is in its turn parallel to the face MN; that is,

Every vertical plane is conjugate to the free surface.

Now, as we have selected the faces of MNLK, the pressure on the faces parallel to the paper when drawn parallel to the free surface will be horizontal, so that the stress normal to the paper is *a principal stress,* and the plane of the paper is the plane of the other two principal stresses. We can apply therefore our preceding results.

Let r' be the stress on the vertical faces MK and NL: it must be parallel to the free surface, and so its direction is that of the sloping layer, so that every point in that layer is in the same state of strain, and r' is transmitted along the layer to act on the wall.

To find out the ratio of the *pair of conjugate stresses* r and r' whose common obliquity is γ.

From particular case (a) of the inverse problem (p. 85), we have—

$$\sqrt{\left\{\frac{(r+r')^2}{4\cos^2\gamma} - rr'\right\}} = \frac{p-q}{2} \text{..................(B)}$$

and

$$\frac{r+r'}{2\cos\gamma} = \frac{p+q}{2} \text{..................(A)}$$

squaring both,

we have
$$\frac{(r+r')^2}{4\cos^2\gamma} - rr' = \frac{(p-q)^2}{4}$$

and
$$\frac{(r+r')^2}{4\cos^2\gamma} = \frac{(p+q)^2}{4}$$

dividing,

we have
$$1 - \frac{4rr'\cos^2\gamma}{(r+r')^2} = \left(\frac{p-q}{p+q}\right)^2.$$

But when earth is just in equilibrium,

$$\frac{p}{q} = \frac{1 + \sin\phi}{1 - \sin\phi},$$

or

$$\frac{p - q}{p + q} = \sin\phi.$$

$$\therefore\quad 1 - \frac{4rr'\cos^2\gamma}{(r + r')^2} = \sin^2\phi.$$

$$\therefore\quad \frac{4rr'\cos^2\gamma}{(r + r')^2} = 1 - \sin^2\phi$$

$$= \cos^2\phi.$$

$$\therefore\quad \frac{4rr'}{(r + r')^2} = \frac{\cos^2\phi}{\cos^2\gamma},$$

or

$$(r + r')^2 = 4rr'\frac{\cos^2\gamma}{\cos^2\phi} \quad \text{.........................I.}$$

Now

$$4rr' = 4rr'.$$

Subtracting,

$$(r - r')^2 = 4rr'\left(\frac{\cos^2\gamma}{\cos^2\phi} - 1\right)$$

$$= 4rr'\frac{\cos^2\gamma - \cos^2\phi}{\cos^2\phi} \quad \text{...............II.}$$

Dividing II. by I.

$$\left(\frac{r - r'}{r + r'}\right)^2 = \frac{\cos^2\gamma - \cos^2\phi}{\cos^2\phi} \cdot \frac{\cos^2\phi}{\cos^2\gamma}$$

$$= \frac{\cos^2\gamma - \cos^2\phi}{\cos^2\gamma}.$$

$$\therefore\quad \frac{r - r'}{r + r'} = \frac{\pm\sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma}.$$

On both sides add numerator to denominator for a new numerator, and subtract numerator from denominator for a new denominator.

$$\therefore\quad \frac{r}{r'} = \frac{\cos\gamma \pm \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma \mp \sqrt{(\cos^2\gamma - \cos^2\phi)}}.$$

That is, r may be greater than r' in the ratio taken with the upper signs, when the earth is on the point of spreading, and the wall is subjected to the least possible value of r', and again r may be less than r' in the ratio taken with the lower signs when the wall is artificially pressed against the earth till the earth is on the point of heaving up and the wall subjected to the greatest possible value of r'.

For equilibrium of retaining wall take upper signs, and reverse the proposition,

$$\frac{r'}{r} = \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}}.$$

COR.—In extreme case $\gamma = \phi$, and $\therefore \frac{r'}{r} = 1$, or $r' = r$, *i.e.*, the conjugate thrusts are equal.

Substituting the value of r, we have the least intensity of the conjugate thrust at the depth D,

$$r' = w\,\mathrm{D}\cos\gamma\,\frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}},$$

and its direction is parallel to the upper free surface.

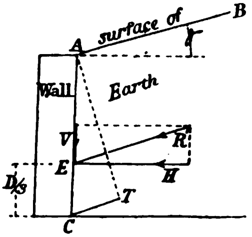


Fig. 60.

On right hand side of equation everything is constant but D, so that r' varies as the depth.

Let D be depth of vertical face of wall. Lay off CT to represent r'. Join AT, and the arrows will represent the thrust on the wall. The average intensity is $\frac{r'}{2}$, and the total thrust is

$$\mathrm{R}' = \text{av. inten.} \times \text{area exposed}$$

$$= \frac{r'}{2}\ \text{lbs. per sq. ft.} \times \mathrm{D}\ \text{sq. ft.}$$

$$= \frac{w\mathrm{D}^2}{2}\cos\gamma\,\frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}}\ \text{lbs.}$$

and it passes through the centre of gravity of the triangle ATC or through the point E where $\mathrm{EC} = \frac{\mathrm{D}}{3}$ and parallel to BA.

Resolving R′ into horizontal and vertical components,

$$\mathrm{H} = \mathrm{R}'\cos\gamma,$$
$$\mathrm{V} = \mathrm{R}'\sin\gamma,$$

H tends to make the wall slide as a whole along the bed joint at C, and for equilibrium of the wall, weight of wall × coefficient of friction at bed joint must be greater than H.

H tends to overturn the wall with a moment

$$\mathrm{M} = \mathrm{H}\left(\frac{\mathrm{D}}{3}\right)$$

$$= \frac{w\mathrm{D}^3}{6}\cos^2\gamma\,\frac{\cos\gamma - \surd(\cos^2\gamma - \cos^2\phi)}{\cos\gamma + \surd(\cos^2\gamma - \cos^2\phi)}\text{ ft.-lbs.}$$

For equilibrium of wall, its weight multiplied by KC feet must exceed M.

NOTE.—V, the tangential component of the pressure of earth on the back of wall multiplied by KC, tends to resist M and to increase effective weight of wall, but the friction of the earth there is liable to be destroyed by water lodging, and it is not safe to rely on it.

Examples.

86. A wall 9 ft. high faces the steepest declivity of earth at a slope of 20° to the horizon; weight of earth 130 lbs. per cub. ft., angle of repose 30°. Find average intensity of thrust in wall, the total thrust on wall, the horizontal component of thrust, and the overturning moment of this component.

Data,—$\gamma = 20^\circ$, greatest slope of earth,
$\phi = 30^\circ$, angle of repose of earth,
$w = 130$ lbs. wt. of cub. ft. of earth,
$\mathrm{D} = 9$ ft. depth of wall.

r = int. of vert. stress at depth 9 ft. per sq. ft. of sloping surface,

$= w\text{D} \cos \gamma,$

$= 130 \times 9 \times \cos 20° = 1099$ lbs. per sq. in.

r' is the conjugate thrust whose direction is parallel to sloping surface when earth is just about to spread.

$$\frac{r'}{r} = \frac{\cos \gamma - \surd(\cos^2\gamma - \cos^2\phi)}{\cos \gamma + \surd(\cos^2\gamma - \cos^2\phi)}$$

$$= \frac{\cdot 9397 - \cdot 364}{\cdot 9397 + \cdot 364} = \cdot 442.$$

$\therefore \quad r' = 1099 \times \cdot 442 = 486$ lbs. per sq. ft.

Aver. int. of conj. stress = 243 „ „

Total thrust per lin. ft. of wall

R' = aver. int. × area

= 243 lbs. per sq. ft. × 9 sq. ft. = 2187 lbs.

$\text{H} = \text{R}' \cos \gamma = 2055$ lbs.

$\text{M} = \text{H}$ lbs. $\times \frac{\text{D}}{3}$ ft. = 6165 ft.-lbs.

Weight of wall multiplied by coefficient of friction at lowest bed joint (if horizontal) must equal H multiplied by a factor of safety. Weight of wall multiplied by $\frac{3}{8}$ths of half thickness at bottom must equal M multiplied by factor of safety. (Weight in lbs., thickness in feet.)

87. The slope of a cutting being one in one and a half; weight of earth being 120 lbs. per cub. ft., and its angle of repose 36°. Find average intensity, amount of horizontal component, and overturning moment of the thrust upon a 3 ft. retaining wall at bottom of slope.

$\tan \gamma = \frac{1}{1\cdot 5} = \cdot 6666. \quad \therefore \gamma = 33° \ 42'.$

$\phi = 36°$ and $w = 120$ lbs.

$\text{D} = 3$ ft. $\quad \therefore r = w\,\text{D} \cos \gamma = 299$ lbs. per sq. ft.

$$\frac{r'}{r} = \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} = \cdot 62.$$

$$\therefore \quad r' = 299 \times \cdot 62 = 185\cdot 4,$$

and aver. int. of stress = 92·7 lbs. per sq. ft.

$$\text{R}' = \frac{r'}{2} \times \text{D sq. ft.} = 278 \text{ lbs.}$$

$$\text{H} = \text{R}' \cos\gamma = 231\cdot 6 \text{ lbs.}$$

$$\text{M} = \text{H} \times \frac{\text{D}}{3} = 231\cdot 6 \text{ ft.-lbs.}$$

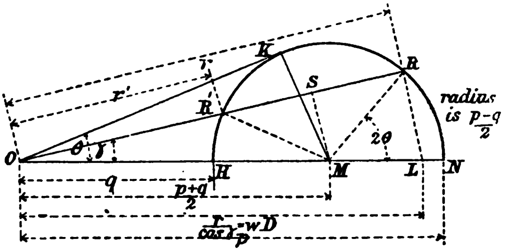


Fig. 61.

Geometrically, $r = w\text{D} \cos\gamma$, being the vertical conjugate thrust, on a layer at depth D, due to the weight of the earth, to find in terms of r,

r', the conjugate thrust parallel to layer.

p and q, the principal stresses in the plane of paper.

θ, the inclination to the direction of r (i.e., the vertical), of the axis of p.

And the third principal stress normal to plane of paper.

Construction. Let $\text{OM} = \frac{p+q}{2}$

Make MOK = ϕ, the angle of repose of earth,

Drop MK perpendicular on OK.

Then $\text{MR} = \frac{p-q}{2}$. case (*d*), page 73.

Draw semicircle.

$$\mathrm{OH} = \mathrm{OM} - \mathrm{MH} = q,$$
$$\mathrm{ON} = \mathrm{OM} + \mathrm{MN} = p.$$

Draw OR′R, making NOR $= \gamma$, the common obliquity of the conjugate thrusts r and r', and

$$\left.\begin{array}{l}\mathrm{OR} = r \\ \mathrm{OR}' = r'\end{array}\right\} \qquad \text{Case } (a), \text{ page 84.}$$

The relations among those are easily expressed trigonometrically by supposing OM proportional to unity, when

OM prop. to 1.
radius ρ „ $\sin\phi$.
OS „ $\cos\gamma$.
MS „ $\sin\gamma$.
$\mathrm{RS} = \sqrt{(\mathrm{MR}^2 - \mathrm{MS}^2)}$ or $\sqrt{(\rho^2 - \mathrm{MS}^2)}$ (Euc. I. 47.)

$$\begin{array}{ll}\text{prop. to } \sqrt{(\sin^2\phi - \sin^2\gamma)} & \left\{\begin{array}{l}\text{put } (1-\cos^2\phi) \text{ for } \sin^2\phi \\ \text{and } (1-\cos^2\gamma) \text{ for } \sin^2\gamma\end{array}\right. \\ \quad\text{„} \quad \sqrt{(\cos^2\gamma - \cos^2\phi)} & \end{array}$$

p or ON $=$ OM $+ \rho$
prop. to $(1 + \sin\phi)$,

q or OH $=$ OM $- \rho$
prop. to $(1 - \sin\phi)$,

r' or OR′ $=$ OS $-$ RS
prop. to $\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}$,

and r or OR $=$ OS $+$ RS
prop. to $\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}$.

$$\therefore \quad \left\{\begin{array}{l} \dfrac{r'}{r} = \dfrac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}}, \\[2ex] \dfrac{p}{r} = \dfrac{1 + \sin\phi}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} \\[2ex] \dfrac{q}{r} = \dfrac{1 - \sin\phi}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}}. \end{array}\right.$$

The axis of p makes an angle $\theta = \frac{1}{2}$ RMN, with ON the normal to the (sloping layer) plane upon which r acts and on the same side.

Also $$\cos 2\theta = \frac{2r \cos \gamma - p - q}{p - q}.\quad \text{Case } (a) \text{ page } 85.$$

Since the earth is upon the point of spreading, the principal stress normal to the paper will be the least possible, that is, it will be equal to q.

Hence this is the horizontal stress on vertical face of a wall running up the steepest declivity: that it is greater than the horizontal thrust for horizontal layers may be seen by supposing the figure on page 99 to be drawn to such a scale that the line ON (which there represents the weight of

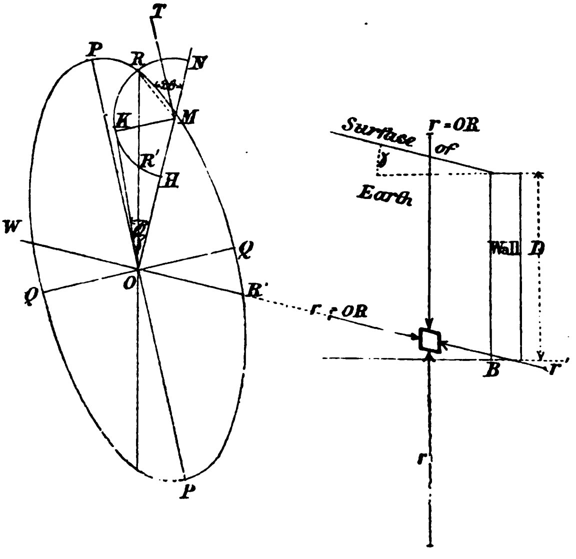


Fig. 62.

vertical column of earth) will be of the same length as OL (which in this case represents same), and superimposing it upon this figure, OH there will be seen to be shorter than OH here.

From a point O on layer at depth D draw ON the normal to layer. Lay off OM, etc., complete construction as in last.

Draw OP parallel to MT the bisector of RMN; this and OQ are the axes of the ellipse of stress parallel to plane of paper.

Lay off OP = ON and OQ = OH, and draw ellipse; since MR is always less than OM for like principal stresses, MRO $> \gamma$, $\therefore$ NMR $> 2\gamma$, $\therefore \theta > \gamma$, and OP is always in the acute angle ROW between the vertical and the line of greatest declivity, and making $(\theta - \gamma)$ with vertical.

Since the third principal stress normal to paper is also OQ, then if the ellipse revolves about PP′ it will sweep out a *spheroid.*

The resultant stress upon *any* plane is *some* vector of this spheroid. We know that for vertical planes the stress, being conjugate to r, is always in the sloping plane WB, so that all the vectors of spheroid lying in that plane determine the stresses upon all vertical planes both in *direction* and intensity.

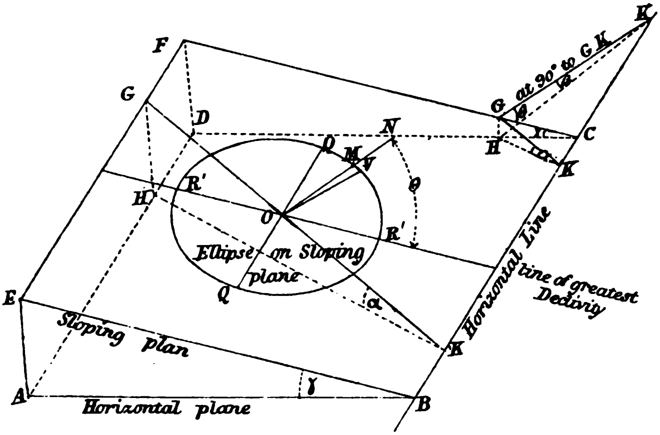


Fig. 63.

The trace of the spheroid upon the sloping plane EBCF is an ellipse whose major axis is OR′ = r', and minor axis the third principal thrust equal to OQ = q; and it lies on the sloping plane with its minor axis horizontal and the major on line of greatest declivity. The thrust on any vertical plane is the vector of this ellipse which is conjugate to it.

Thus a wall with vertical face whose found runs along GK (a line on the sloping surface inclined at an angle α to the horizon) sustains a thrust represented in direction and intensity by the vector OV which is conjugate to GOK.

ON is a line on the sloping surface at right angles to GK. The component of OV in the direction ON is that which is effective, the other being tangential to wall. As on page 65, construct OMV with V in place of R and r' in place of p, and

$$\text{effective comp. of OV} = r' \cos^2\theta + q \sin^2\theta,$$

and this is itself inclined to normal to face of wall at an angle β.

$$\therefore \text{ Hor. thrust on wall} = (r' \cos^2\theta + q \sin^2\theta) \cos \beta.$$

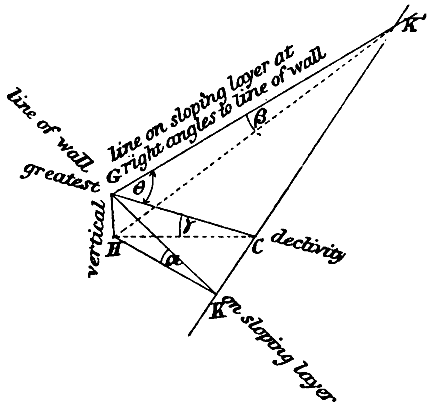


Fig. 64.

θ and β are determined thus:

Let GKH be a vertical plane, and firstly, let GH be vertical. Let also GCH be a vertical plane perp. to KCK′. Then

$$\sin\theta = \cos \mathrm{KGC} = \frac{\mathrm{GC}}{\mathrm{GK}} = \frac{\mathrm{GH}\ \mathrm{cosec}\ \gamma}{\mathrm{GH}\ \mathrm{cosec}\ \alpha} = \frac{\sin\alpha}{\sin\gamma}$$

gives θ.

Secondly, let GH be perp. to GK, and let HK′ be perp. to the plane GKH. HK′ is therefore a line in the horizontal plane KHK′.

$$\begin{aligned}\therefore\ \text{KK}'^2 &= \text{HK}^2 + \text{HK}'^2 && \because\ \text{KHK}' \text{ is a right angle.}\\ &= \text{GK}^2 + \text{GH}^2 + \text{HK}'^2 && \because\ \text{KGH} \quad ,, \quad ,,\\ &= \text{GK}^2 + \text{GK}'^2 && \because\ \text{GHK}' \quad ,, \quad ,,\end{aligned}$$

$\therefore$ KGK$'$ is a right angle.

Hence it is evident that $\text{GK}'\text{H} = \beta$,

$$\therefore\quad \sin\beta = \frac{\text{GH}}{\text{GK}'} = \frac{\text{GK}\tan a}{\text{GK}\tan \text{GKK}'} = \frac{\tan a}{\tan\theta}$$

gives β.

Examples.

88. A cutting having 3 ft. retaining walls is made on ground sloping at 20° to the horizon. Weight of earth is 120 lbs. per cub. ft. and its angle of repose 30°. Find the horizontal thrust and the overturning moment—1st, When cutting runs horizontal; 2nd, When cutting runs up steepest declivity; and 3rd, When cutting runs up at a declivity of 15° to the horizon.

Data.—$\text{D} = 3$ ft. $\qquad \gamma = 20°.$

$w = 120$ lbs. $\qquad \phi = 30°.$

(1*st*) $r = w\text{D}\cos\gamma = 338$ lbs. per sq. ft.

= stress on slop. layer at depth D, being vertical.

$$\frac{r'}{r} = \frac{\cos\gamma - \sqrt{(\cos^2\gamma - \cos^2\phi)}}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} = \frac{\cdot 576}{1\cdot 304} = \cdot 442.$$

$\therefore\ r' = 338 \times \cdot 442 = 149\cdot 4$ lbs. per sq. ft.

= conj. stress on vertical face of wall, being in sloping layer inclined at γ.

$r'\cos\gamma = 140\cdot 4$ lbs. per sq. ft.

= horizontal thrust on wall, at foot of wall.

Aver. do. = 70·2.

Total do. = aver. int. × area

= 70·2 × 3 = 210·6 lbs. per lineal ft. of wall.

$$\text{Moment} = 210\cdot 6 \times \frac{\text{D}}{3} = 210\cdot 6 \text{ ft.-lbs.}$$

(*2nd*) $$\frac{q}{r} = \frac{1 - \sin\phi}{\cos\gamma + \sqrt{(\cos^2\gamma - \cos^2\phi)}} = \frac{\cdot 5}{1{\cdot}304} = {\cdot}383.$$

$\therefore\ q = 338 \times {\cdot}383 = 129{\cdot}4$ lbs. per sq. ft.

= least principal stress in sec. on greatest declivity

= also third principal stress which is horizontal on face of vertical wall.

Aver. do. = 65 lbs. per sq. ft.

Total do. = 65 × area = 65 × 3 = 195 lbs. per lin. ft. of wall.

Moment $= 195$ lbs. $\times \frac{\mathrm{D}}{3} = 195$ ft.-lbs.

(*3rd*) Section of spheroid of stress on the sloping layer is an ellipse whose axes are r' and q. OV is the thrust conjugate to vertical plane.

$$a = 15^\circ.$$

Now $$\sin\theta = \frac{\sin a}{\sin\gamma},$$

$$\begin{aligned} \text{L.}\sin\theta &= \text{L.}\sin.\ a - \text{L.}\sin\gamma + 10 \\ &= 9{\cdot}4129962 - 9{\cdot}5340517 + 10 \\ &= 9{\cdot}8789445. \\ \therefore\ \theta &= 49^\circ\ 11'. \end{aligned}$$

Also $$\sin\beta = \frac{\tan a}{\tan\theta} = \frac{\tan 15^\circ}{\tan 49^\circ\ 11'}.$$

$$\begin{aligned} \therefore\ \text{L.}\sin\beta &= 10 + \text{L.}\tan 15^\circ - \text{L.}\tan 49^\circ\ 11' \\ &= 10 + 9{\cdot}4280525 - 10{\cdot}0636448 \\ &= 9{\cdot}3644077. \\ \therefore\ \beta &= 13^\circ\ 23'. \end{aligned}$$

* Effective com. of OV $= r' \cos^2\theta + q \sin^2\theta$

$= 63{\cdot}83 + 74{\cdot}11$

$= 137{\cdot}94$ lbs. per sq. ft.

= thrust on vertical face of wall along OMN the intersection of sloping layer and vertical plane at right angles to face of wall.

This is inclined to face of wall at an angle β.

$\therefore$ Hor. thrust $= 137{\cdot}94 \cos\beta$

$= 134{\cdot}2$ lbs. per sq. ft.

Aver. do. $= 67{\cdot}1$ „ „

Total do. $= 67{\cdot}1 \times$ area $= 201{\cdot}3$ lbs. per lin. ft. of wall

Moment $= 201{\cdot}3 \times \frac{D}{3} = 201{\cdot}3$ ft.-lbs.

---

* The effective component of OV can be more easily calculated by the formula

$$\text{Effective component of OV} = \frac{r' + q}{2} + \frac{r' - q}{2} \cos 2\theta$$

$$= 139{\cdot}4 + 10(-{\cdot}146)$$

$$= 139{\cdot}4 - 1{\cdot}46$$

$$= 137{\cdot}94.$$

END.

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